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Generalised N=2 permutation branes

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Abstract: Generalised permutation branes in products of $N = 2$ minimal models play an important role in accounting for all RR charges of Gepner models. In this paper an explicit conformal field theory construction of these generalised permutation branes for one simple class of examples is given. We also comment on how this may be generalised to the other cases.

Keywords: Conformal Field Models in String Theory, D-branes.

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1. Introduction

D-branes on Calabi–Yau manifolds play a central role for many phenomenologically interesting string compactifications, and it is important to understand them in detail. In the large volume regime D-branes can be successfully described geometrically, but this approach generically breaks down when the size of the compactification manifold is of the order of the string length. In this regime a microscopic formulation in terms of conformal field theory is required.

At specific points in the moduli space of small Calabi–Yau manifolds an explicit conformal field theory description is known. The relevant constructions include, in particular, toroidal orbifold constructions, and Gepner models [1]. Gepner models are orbifolds of tensor products of $N = 2$ minimal models. They can be obtained as the IR limit of Landau–Ginzburg models [2 – 5]. Via the gauged linear sigma-model, Landau–Ginzburg models are in turn directly related to Calabi–Yau manifolds [6].

A certain class of supersymmetric D-branes for Gepner models can be relatively easily constructed: these are the so-called tensor product or Recknagel–Schomerus (RS) D-branes [7] that preserve the different $N = 2$ superconformal algebras separately. Their geometrical interpretation was understood in [8]. A slight generalisation of the RS construction involves D-branes that preserve the different $N = 2$ superconformal algebras up to a permutation, the so-called permutation branes [9]. Their geometrical interpretation has been clarified in [10]. In general, however, these two classes of constructions do not account for all the RR charges; for example, for the A-type Gepner models, there are 31 models (of 147) for which this is not the case [11]. It is therefore an important open problem to find the conformal field theory constructions that account for these missing charges. In this paper we shall give a partial answer to this problem by constructing the missing branes for one class of examples (that deals with 3 of the 31 cases).

The new development that has sparked progress in this area is the recent characterisation of B-type D-branes for Landau–Ginzburg models in terms of matrix factorisations of the superpotential. This proposal, which was first made by Kontsevich in unpublished work, has been supported (and physically motivated) in a number of papers [12 – 16]. It has subsequently been applied to the analysis of D-branes on Calabi–Yau manifolds in [17 – 19, 10, 20]. In particular, the factorisations corresponding to RS and permutation branes could be identified.

Using these techniques it was shown in [11] that the remaining RR charges in Gepner models can all be accounted for in terms of factorisations that are generalisations of the ordinary permutation factorisations. Furthermore, together with the usual RS and permutation constructions, they account for a basis of the complete quantised charge lattice.

These new factorisations are based on writing the superpotential for two minimal models of levels k_i as a product

$$W = x^{r_1 d} + y^{r_2 d} = \prod_\xi (x^{r_1} - \xi y^{r_2}) \,, \tag{1.1}$$

where $d = \gcd\{k_i+2\}$, $k_i+2 = r_i d$, and ξ runs through the d^{th} roots of -1. If $r_1 = r_2 (= 1)$, we have the ordinary permutation case, and for $d = 2$ the corresponding branes are the resolved tensor product branes that occur when both levels are even [11]. In all other cases, however, the corresponding branes must describe new constructions that go beyond the known class of maximally symmetric branes. These 'generalised permutation branes' thus arise when the shifted levels of the two factor theories are different, but have a non-trivial common factor $d \geq 3$.

In this paper we shall give an explicit conformal field theory construction of the generalised permutation branes for the simplest example, namely for the product of two $N = 2$ minimal models with levels $k = 1$ and $k = 4$.[1] This case is particularly simple since the total central charge of the theory is $c = 3$. For $c = 3$ the representation theory of the $N = 2$ algebra has been studied in detail [21], and (for free theories) the most general $N = 2$ superconformal branes are known [22]. We can thus analyse the above product theory in terms of the diagonal $N = 2$ algebra at $c = 3$, and determine the complete set of $N = 2$ Ishibashi states. Using various symmetry constraints, in particular the invariance of the boundary states under spectral flow, we can then construct them explicitly. The resulting branes reproduce precisely the topological open string spectrum that is predicted by the matrix factorisation approach.

The above product theory at $c = 3$ is in fact equivalent to a $\mathbb{Z}_6$-orbifold of a torus theory [23]. The generalised permutation branes (P-branes) we constructed above should therefore also have a description in terms of the torus orbifold, and this is indeed the case. Among other things, this identification confirms that the boundary states for the generalised permutation branes we have constructed in terms of the diagonal $N = 2$ theory are consistent. It also allows us to check at least some of the factorisation constraints for them. Finally, it follows from this analysis that the generalised permutation branes for the $(k = 1) \times (k = 4)$ theory preserve in fact a large W-algebra, not just the diagonal $N = 2$ algebra. However, as we shall explain, this property seems to be particular to the $(k = 1) \times (k = 4)$ case, and does not appear to generalise. In fact, our arguments suggest that for generic levels the tensor product of two $N = 2$ algebras does not contain any non-trivial W-algebra extension of the diagonal N=2 subalgebra.

There is a second natural class of branes for the $(k = 1) \times (k = 4)$ theory that could have deserved the name 'generalised permutation branes'. These $\hat{P}$-branes also preserve a large (but different) W-symmetry. Their spectrum is however different from the P-branes, and they correspond to a different class of matrix factorisations that we identify. We also show that the $\hat{P}$-branes have a very simple description in terms of the permutation orbifold of the $(k = 1)^{\times 3}$ theory (that is equivalent to the $(k = 1) \times (k = 4)$ theory).

[1] The shifted levels $1 + 2 = 3$ and $4 + 2 = 6$ contain then 3 as a common factor.

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The paper is organised as follows. In section 2 we give the explicit construction of the P-branes for the example of $(k=1)\times(k=4)$, and show that their topological spectrum agrees with the predictions from the matrix factorisation point of view. In section 3, the description of the P-branes in terms of the torus orbifold is given, and the factorisation constraints are analysed. We also explain the extended W-symmetry that is preserved by these branes, and why this property seems to be specific to the example of $(k=1)\times(k=4)$. Section 4 is devoted to the analysis of the $\hat{P}$-branes, and section 5 contains our conclusions. We have included a number of appendices in which some of the more technical details of our constructions are explained.

2. The $A_1 \times A_4$ model

The simplest example where generalised permutation branes appear is the tensor product of two A-type $N=2$ minimal models with levels $k=1$ and $k=4$, respectively. In the Landau–Ginzburg description this theory is given by the superpotential

$$W = x^3 + y^6\,, \tag{2.1}$$

whereas in conformal field theory it corresponds to the spectrum

$$\bigoplus_{[l_i,m_i,s_i]} \Big(\mathcal{H}_{[l_1,m_1,s_1]}\otimes\mathcal{H}_{[l_2,m_2,s_2]}\Big)\otimes\Big(\bar{\mathcal{H}}_{[l_1,m_1,-s_1]}\otimes\bar{\mathcal{H}}_{[l_2,m_2,-s_2]} \oplus\bar{\mathcal{H}}_{[l_1,m_1,-s_1+2]}\otimes\bar{\mathcal{H}}_{[l_2,m_2,-s_2+2]}\Big)\,. \tag{2.2}$$

Here the sum runs only over those pairs of representations for which s_1-s_2 is even, and $[l_1,m_1,s_1]$ ($[l_2,m_2,s_2]$) denotes the representations of the bosonic subalgebra of the $N=2$ superconformal algebra at $k=1$ ($k=4$); for a description of the usual conventions see appendix A (compare also [7, 24, 10]).

2.1 Branes in matrix factorisations

Let us first briefly review what information about the generalised permutation branes can be obtained from the matrix factorisation approach. We begin by collecting some basic facts about the relation between matrix factorisations and D-branes.

According to Kontsevich's proposal, B-type D-branes in Landau–Ginzburg models correspond to matrix factorisations of the superpotential W,

$$E\,J = J\,E = W\cdot\mathbf{1}\,, \tag{2.3}$$

where E and J are $r\times r$ matrices with polynomial entries in the superfields. This condition can be more succinctly written as

$$Q^2 = W\cdot\mathbf{1}\,,\qquad \text{where}\qquad Q=\begin{pmatrix}0 & J\\ E & 0\end{pmatrix}\,. \tag{2.4}$$

The matrices E and J describe boundary fermions whose presence is required to cancel the supersymmetry variation of the bulk F-term in the presence of a boundary. This approach was proposed in unpublished form by Kontsevich, and the physical interpretation of it was given in [12 – 14]; for a good review of this material see for example [25].

Given the close relation between Landau–Ginzburg models and $N=2$ superconformal field theories, we therefore expect that there is a one-to-one correspondence between matrix factorisations and $N=2$ superconformal D-branes. Furthermore, the topological part of the open string spectrum between any two such D-branes should correspond to a suitable cohomology that can be directly calculated in terms of the matrix factorisation description. This will allow us to match matrix factorisations with superconformal D-branes in conformal field theory.

For the case of the above Landau–Ginzburg model we have the obvious tensor product factorisations

$$E = \begin{pmatrix} E_1 & E_2 \\ J_2 & -J_1 \end{pmatrix} , \qquad J = \begin{pmatrix} J_1 & E_2 \\ J_2 & -E_1 \end{pmatrix} , \tag{2.5}$$

where $E_1 = x^l$, $J_1 = x^{3-l}$, $E_2 = y^m$, $J_2 = y^{6-m}$. Up to the usual equivalences we may restrict ourselves to $l=1$ and $m=1,2,3$.

These factorisations however do not carry any RR charge. The simplest factorisation that carries RR charge is the generalised permutation factorisation that was first considered in [11]. In order to see how it arises one observes that W can be written as

$$W = (x - \xi_0 y^2)\,(x - \xi_1 y^2)\,(x - \xi_2 y^2)\,, \qquad \xi_j = e^{\frac{(2j+1)i\pi}{3}}\,. \tag{2.6}$$

A rank 1 factorisation of $W = EJ$ is then obtained by taking E to be one of the three factors above, while J is the product of the other two. There are three such choices that we shall denote by Q_ξ (where ξ denotes the root that appears in the factor E), and together with their reverse factorisations Q^r_ξ (where the roles of E and J are interchanged) we have in total six such factorisations.

It is the aim of this paper to identify the corresponding branes (that we shall call 'P-branes' in the following) in conformal field theory. Before we start to construct the superconformal boundary states we shall collect some information on the spectrum of these D-branes from the matrix factorisation point of view. This information will help us identify these branes correctly in conformal field theory.

2.1.1 Spectra from matrix factorisations

It is straightforward to calculate the topological open string spectrum between any two such factorisations. By definition, the spectrum between two factorisations Q_1 and Q^r_2 can be obtained from that between Q_1 and Q_2 by exchanging the roles of the bosons and fermions. It is therefore sufficient to give the results only for the spectrum between the factorisations Q_ξ. For the case at hand, the results of [11] can be summarised as follows:

1. The self-spectrum of Q_ξ with itself contains no fermions and four bosons of U(1) charge $0, \frac{1}{3}, \frac{2}{3}, 1$.

2. The spectrum between Q_ξ and $Q_{\xi'}$, where $\xi \neq \xi'$, contains no bosons, and two fermions of U(1) charge $\frac{1}{3}, \frac{2}{3}$.

3. The relative spectrum between the tensor product factorisations corresponding to $E_1 = x$, $E_2 = y$ and Q_ξ is independent of ξ. It contains one boson of $\mathrm{U}(1)$ charge $\frac{2}{3}$ and one fermion of $\mathrm{U}(1)$ charge $\frac{1}{3}$.

After these preparations we shall now construct the corresponding boundary states in conformal field theory.

2.2 Branes in conformal field theory

The branes that correspond to matrix factorisations of the Landau–Ginzburg model are superconformal B-type D-branes of the diagonal $N = 2$ algebra. The boundary states $\|B\rangle\!\rangle$ of B-type D-branes are characterised by the condition that

$$\begin{aligned} \left(L_n - \tilde{L}_{-n}\right) \|B\rangle\!\rangle &= 0 \\ \left(J_n + \tilde{J}_{-n}\right) \|B\rangle\!\rangle &= 0 \\ \left(G_r^{\pm} + i\,\eta\,\tilde{G}_{-r}^{\pm}\right) \|B\rangle\!\rangle &= 0\,, \end{aligned} \tag{2.7}$$

where $\eta = \pm$ distinguishes the two different spin-structures. In the following we shall work with one fixed choice of η, say $\eta = +1$.

The theory in question possesses however more symmetry since we have a tensor product of two $N = 2$ algebras with $c = 1$ $(k = 1)$ and $c = 2$ $(k = 4)$, respectively. One simple class of branes are those that respect the full chiral symmetry

$$\begin{aligned} \left(L_n^{(1)} - \tilde{L}_{-n}^{(1)}\right) \|B\rangle\!\rangle = \left(L_n^{(2)} - \tilde{L}_{-n}^{(2)}\right) \|B\rangle\!\rangle &= 0 \\ \left(J_n^{(1)} + \tilde{J}_{-n}^{(1)}\right) \|B\rangle\!\rangle = \left(J_n^{(2)} + \tilde{J}_{-n}^{(2)}\right) \|B\rangle\!\rangle &= 0 \\ \left(G_r^{(1)\pm} + i\,\tilde{G}_{-r}^{(1)\pm}\right) \|B\rangle\!\rangle = \left(G_r^{(2)\pm} + i\,\tilde{G}_{-r}^{(2)\pm}\right) \|B\rangle\!\rangle &= 0\,. \end{aligned} \tag{2.8}$$

These 'tensor product branes' (or Recknagel-Schomerus branes [7]) are labelled by a quadruple (L_1, L_2, S_1, S_2) of integers where $L_1 = 0, 1$ and $L_2 = 0, \ldots, 4$, and S_1, S_2 are defined modulo 4. The difference $S_1 - S_2$ is always even and there are the identifications

$$(L_1, L_2, S_1, S_2) \sim (1 - L_1, L_2, S_1 + 2, S_2) \sim (L_1, 4 - L_2, S_1, S_2 + 2) \sim (L_1, L_2, S_1 + 2, S_2 + 2) \tag{2.9}$$

among the quadruples, so we can always choose $L_1 = 0$ and $L_2 = 0, 1, 2$. The choice $\eta = +1$ in (2.7) corresponds now to considering only branes with even S_1 and S_2. Given the above identifications, there are only two cases, namely $S_1 = S_2 = 0$ and $S_1 = 0, S_2 = 2$, which are anti-branes of one another. It is therefore sufficient to give only the branes $\|T(L_1, L_2)\rangle\!\rangle$ with $S_1 = S_2 = 0$, for which we have

$$\|T(0,0)\rangle\!\rangle = \frac{3^{1/4}}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}\rangle\!\rangle + \sqrt{2}|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{3}{2}\tfrac{3}{2}\rangle\!\rangle + |\tfrac{1}{2}\tfrac{1}{2}, \tfrac{5}{2}\tfrac{5}{2}\rangle\!\rangle\right) + \sqrt{\frac{3}{2}}\left(|1\,1, 1\,1\rangle\!\rangle + |1\,1, 2\,2\rangle\!\rangle\right) \tag{2.10}$$

$$\|T(0,1)\rangle\!\rangle = \frac{3^{\frac{3}{4}}}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}\rangle\!\rangle - |\tfrac{1}{2}\tfrac{1}{2}, \tfrac{5}{2}\tfrac{5}{2}\rangle\!\rangle\right) + \sqrt{\frac{3}{2}}\left(|1\,1, 1\,1\rangle\!\rangle - |1\,1, 2\,2\rangle\!\rangle\right) \tag{2.11}$$

$$\|T(0,2)\rangle\!\rangle = 3^{\frac{1}{4}}\left(\sqrt{2}|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}\rangle\!\rangle - |\tfrac{1}{2}\tfrac{1}{2}, \tfrac{3}{2}\tfrac{3}{2}\rangle\!\rangle + \sqrt{2}|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{5}{2}\tfrac{5}{2}\rangle\!\rangle\right)\,. \tag{2.12}$$

Note that $T(0,2)$ is its own anti-brane and does not couple to the RR sector. The states that appear on the right hand side are the B-type tensor product Ishibashi states. Our conventions for the labelling of the representations of the $N=2$ superconformal algebra are explained in appendix A.

These tensor product branes correspond to the tensor product factorisations that were described at the beginning of section 2.1 [13, 14, 26]. In fact, the factorisations with $E_1 = x$, $E_2 = y^m$ correspond to $\|T(0, m-1)\rangle\rangle$.

2.2.1 Generalised permutation branes and spectral flow

The tensor product branes are the only branes that preserve the full chiral symmetry. In particular, since the central charges of the two $N=2$ minimal models are different, it is not possible to construct the usual 'permutation branes' [9] (see also [27, 28]) that are characterised by the gluing conditions

$$\begin{aligned} \left(L_n^{(1)} - \tilde{L}_{-n}^{(2)}\right) \|B\rangle\rangle = \left(L_n^{(2)} - \tilde{L}_{-n}^{(1)}\right) \|B\rangle\rangle = 0 \\ \left(J_n^{(1)} + \tilde{J}_{-n}^{(2)}\right) \|B\rangle\rangle = \left(J_n^{(2)} + \tilde{J}_{-n}^{(1)}\right) \|B\rangle\rangle = 0 \\ \left(G_r^{(1)\pm} + i\,\tilde{G}_{-r}^{(2)\pm}\right) \|B\rangle\rangle = \left(G_r^{(2)\pm} + i\,\tilde{G}_{-r}^{(1)\pm}\right) \|B\rangle\rangle = 0\ . \end{aligned} \tag{2.13}$$

On the other hand, we have seen that there exists an additional simple class of factorisations, namely the generalised permutation factorisations (2.6) that are very reminiscent of the factorisations that correspond to the usual permutation branes [10]. We should therefore expect that the corresponding boundary states are a simple generalisation of the permutation brane construction.

In order to make progress with this construction we use the fact that the relevant branes must respect the diagonal $N=2$ symmetry. The two individual $N=2$ algebras have $c=1$ and $c=2$, respectively, and thus the diagonal $N=2$ algebra has $c=3$. Its representation theory was analysed in detail in [21, 22]. It is not difficult to decompose the various tensor products of $N=2$ representations in terms of the diagonal $N=2$ algebra; the relevant decompositions are given in appendix B.

Since we are considering the diagonal modular invariant (2.2), we have one B-type Ishibashi state of the diagonal $N=2$ algebra for each $N=2$ representation that appears together with its conjugate representation in the decomposition of the tensor product of the two minimal model representations.[2] In particular, it is therefore clear from the results of appendix B that there are infinitely many B-type Ishibashi states for the diagonal $N=2$ algebra, and thus a continuum of B-type D-branes. We expect that the D-branes that correspond to the generalised permutation factorisations are rather special. In order to identify them we need to use additional constraints. One useful constraint comes from the analysis of the spectral flow symmetry.

It is easy to see that the B-type gluing conditions are compatible with applying the spectral flow automorphism (A.7) for the left- and right-movers with opposite values of t. The spectral flow symmetry of boundary states is intimately connected to the quantisation

[2]For the RR sector one also has to keep track of the constraints imposed by the GSO-projection.

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of the $U(1)$-charge in the open string spectrum. Indeed, if the B-type boundary states $\|B\rangle\rangle$ and $\|B'\rangle\rangle$ are invariant under a spectral flow by t $(-t)$ on the left-(right-)movers, then their overlap satisfies

$$\langle\langle B'\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}} z^{\frac{1}{2}(J_0-\tilde{J}_0)}\|B\rangle\rangle = \langle\langle B'\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}+t^2\frac{c}{6}-t\frac{1}{2}(J_0-\tilde{J}_0)} z^{\frac{1}{2}(J_0-\tilde{J}_0)-t\frac{c}{3}}\|B\rangle\rangle \,. \tag{2.14}$$

By modular transformation, this translates into a relation for the open string partition function,

$$\mathrm{tr}_{\mathcal{H}_{BB'}}\left(\tilde{q}^{L_0-\frac{c}{24}}\tilde{z}^{J_0}\right) = \mathrm{tr}_{\mathcal{H}_{BB'}}\left(\tilde{q}^{L_0-\frac{c}{24}}\tilde{z}^{J_0}e^{-2\pi i t J_0}\right) \,. \tag{2.15}$$

This then implies that the $U(1)$-charges q of open strings between $\|B\rangle\rangle$ and $\|B'\rangle\rangle$ are quantised and satisfy $tq\in\mathbb{Z}$.

From the Landau–Ginzburg description we know that the topological part of the open string spectrum between the generalised permutation factorisations only contains $U(1)$-charges which are multiples of $\frac{1}{3}$. This suggests that the corresponding boundary states are in fact invariant under spectral flow by $t=3$. This is further supported by the observation that the tensor product branes $\|T(0,0)\rangle\rangle$ and $\|T(0,2)\rangle\rangle$ are invariant under this spectral flow, and that the topological part of the $\mathrm{U}(1)$-charges in the relative spectrum between these and the generalised permutation factorisations are also quantised in units of $\frac{1}{3}$. The boundary state $\|T(0,1)\rangle\rangle$ on the other hand, picks up a minus sign under the spectral flow by $t=3$, and the relative spectrum between this brane and the generalised permutation branes should therefore only have $\mathrm{U}(1)$-charges satisfying $q\in\frac{1}{6}+\frac{1}{3}\mathbb{Z}$. At least for the topological part of the spectrum this is indeed the case.

The requirement that the boundary states are invariant under spectral flow by $t=3$ restricts their structure significantly. For example, under $t=\pm 3$ the representations of the $k=1$ factor are invariant, but for the $k=4$ theory we exchange

$$t=\pm 3: \qquad \left(\frac{1}{2},\frac{1}{2}\right)\leftrightarrow\left(\frac{5}{2},\frac{5}{2}\right) \qquad \left(\frac{3}{2},\frac{7}{2}\right)\leftrightarrow\left(\frac{9}{2},\frac{1}{2}\right) \qquad \left(\frac{7}{2},\frac{3}{2}\right)\leftrightarrow\left(\frac{1}{2},\frac{9}{2}\right)\,, \tag{2.16}$$

while the representations $(\frac{3}{2},\frac{3}{2})$, $(\frac{5}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{5}{2})$ are invariant (see appendix A). Thus the spectral flow symmetry fixes for example the coefficients of the B-type Ishibashi states that appear in the sector $(\frac{1}{2}\frac{1}{2},\frac{5}{2}\frac{5}{2})$ in terms of those appearing in $(\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2})$, *etc.* Furthermore, since the spectral flow of the $c=3$ theory by $t=\pm 3$ relates the representations of the form $(6n+\frac{5}{2},1)\leftrightarrow(6n+\frac{11}{2},1)$, we also know that the coefficients with which the corresponding Ishibashi states appear must be the same.

There is also a spectral flow symmetry in the open string spectrum which is the result of a charge quantisation in the boundary states. In the theory we are considering, B-type branes can only couple to states with integer $\mathrm{U}(1)$ charge, so that all open string spectra are invariant under a spectral flow by $t=\pm 1$. As this symmetry is a consequence of the B-type condition it does not give additional information on the boundary states, but being aware of this symmetry helps to understand the open string spectra by organising them into spectral flow orbits.

While these considerations constrain the form of the boundary state, they do not determine it uniquely. The construction of the boundary state therefore requires a certain amount of guess work. In the following we shall make an ansatz for the boundary state; we shall then give various pieces of evidence that suggest that this is indeed the correct choice. In particular, we shall show that it reproduces the correct topological open string spectra. In section 3 we shall also show that it has a natural interpretation in terms of a free field construction.

2.2.2 The construction of the *P*-branes in the NSNS sector

We shall first restrict our discussion to the NSNS sector. As we have just explained, the spectral flow symmetry restricts the possible couplings to the different $N = 2$ Ishibashi states. We now make the ansatz that the generalised permutation branes couple to the following linear combinations of Ishibashi states:

$$\begin{aligned} |\tfrac{1}{2}\tfrac{1}{2},\tfrac{1}{2}\tfrac{1}{2}\rangle\!\rangle^P &= |0,0\rangle\!\rangle + \sum_{m=1}^{\infty}\sum_{n=1}^{m} \tfrac{1}{3}\big((-1)^m + (-1)^n + (-1)^{m+n}\big)|m^2+n^2-mn,0\rangle\!\rangle \\ &\quad + \sum_{n=0}^{\infty}\big(|6n+\tfrac{11}{2},1\rangle\!\rangle + |6n+\tfrac{11}{2},-1\rangle\!\rangle\big) \end{aligned} \tag{2.17}$$

$$\begin{aligned} |\tfrac{1}{2}\tfrac{1}{2},\tfrac{5}{2}\tfrac{5}{2}\rangle\!\rangle^P &= \sum_{m=1}^{\infty}\sum_{n=1}^{m} \tfrac{1}{3}\big((-1)^m + (-1)^n + (-1)^{m+n}\big)|m^2+n^2-mn,0\rangle\!\rangle \\ &\quad + \sum_{n=0}^{\infty}\big(|6n+\tfrac{5}{2},1\rangle\!\rangle + |6n+\tfrac{5}{2},-1\rangle\!\rangle\big) \end{aligned} \tag{2.18}$$

$$|\tfrac{1}{2}\tfrac{1}{2},\tfrac{3}{2}\tfrac{3}{2}\rangle\!\rangle^P = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \tfrac{1}{3}\big((-1)^{m+1} + (-1)^n + (-1)^{m+n+1}\big)|\tfrac{1}{3}+(m-n)^2+(m+1)n,0\rangle\!\rangle \tag{2.19}$$

$$|\tfrac{3}{2}\tfrac{1}{2},\tfrac{3}{2}\tfrac{7}{2}\rangle\!\rangle^P = \sum_{\substack{m=1 \\ m \text{ or } n \text{ odd}}}^{\infty}\sum_{n=1}^{m} \tfrac{2}{3}|m^2+n^2-mn,0\rangle\!\rangle \tag{2.20}$$

$$|\tfrac{3}{2}\tfrac{1}{2},\tfrac{9}{2}\tfrac{1}{2}\rangle\!\rangle^P = \sum_{\substack{m=1 \\ m \text{ or } n \text{ odd}}}^{\infty}\sum_{n=1}^{m} \tfrac{2}{3}|m^2+n^2-mn,0\rangle\!\rangle \tag{2.21}$$

$$|\tfrac{1}{2}\tfrac{3}{2},\tfrac{7}{2}\tfrac{3}{2}\rangle\!\rangle^P = \sum_{\substack{m=1 \\ m \text{ or } n \text{ odd}}}^{\infty}\sum_{n=1}^{m} \tfrac{2}{3}|m^2+n^2-mn,0\rangle\!\rangle \tag{2.22}$$

$$|\tfrac{1}{2}\tfrac{3}{2},\tfrac{1}{2}\tfrac{9}{2}\rangle\!\rangle^P = \sum_{\substack{m=1 \\ m \text{ or } n \text{ odd}}}^{\infty}\sum_{n=1}^{m} \tfrac{2}{3}|m^2+n^2-mn,0\rangle\!\rangle \tag{2.23}$$

$$|\tfrac{3}{2}\tfrac{1}{2},\tfrac{1}{2}\tfrac{5}{2}\rangle\!\rangle^P = \sum_{\substack{m=0 \\ m+1 \text{ or } n \text{ odd}}}^{\infty}\sum_{n=0}^{\infty} \tfrac{2}{3}|\tfrac{1}{3}+(m-n)^2+(m+1)n,0\rangle\!\rangle \tag{2.24}$$

$$|\tfrac{1}{2}\tfrac{3}{2},\tfrac{5}{2}\tfrac{1}{2}\rangle\!\rangle^P = \sum_{\substack{m=0 \\ m+1 \text{ or } n \text{ odd}}}^{\infty}\sum_{n=0}^{\infty} \tfrac{2}{3}|\tfrac{1}{3}+(m-n)^2+(m+1)n,0\rangle\!\rangle\ . \tag{2.25}$$

We have chosen here the convention that the combination of $N=2$ Ishibashi states denoted by $|u_1v_1,u_2,v_2\rangle\rangle^P$ describes the B-type Ishibashi states from the representation $(u_1,v_1)\otimes(u_2,v_2)$ of the product theory (for the decomposition of the tensor product representations see appendix B.1). On the right hand side, the $N=2$ Ishibashi states are denoted by their weight and charge (with respect to the diagonal $N=2$ algebra). We have not distinguished the different $N=2$ Ishibashi states with the same weight and charge; for example there are six different Ishibashi states $|m^2+n^2-mn,0\rangle\rangle$ for each allowed combination of m and n that appear in six different sectors. In addition, different combinations of m and n are understood to correspond to different Ishibashi states even if their conformal weight and charge coincide. For the first three sectors (for which we also have tensor product Ishibashi states) we have furthermore chosen the convention that the tensor product Ishibashi states involve all relevant $N=2$ Ishibashi states with (relative) coefficient 1.

With these notations we now claim that the boundary states for the generalised permutation branes are $(j=0,1,2)$

$$\begin{aligned}\|P(j)\rangle\rangle_{\rm NSNS} \;=\; \frac{3^{\frac14}}{\sqrt2}\Big(&|\tfrac12\tfrac12,\tfrac12\tfrac12\rangle\rangle^P+|\tfrac12\tfrac12,\tfrac52\tfrac52\rangle\rangle^P+\sqrt2|\tfrac12\tfrac12,\tfrac32\tfrac32\rangle\rangle^P\\ &+e^{\frac{2\pi i j}{3}}\Big(|\tfrac32\tfrac12,\tfrac32\tfrac72\rangle\rangle^P+|\tfrac32\tfrac12,\tfrac92\tfrac12\rangle\rangle^P+\sqrt2|\tfrac32\tfrac12,\tfrac12\tfrac52\rangle\rangle^P\Big)\\ &+e^{-\frac{2\pi i j}{3}}\Big(|\tfrac12\tfrac32,\tfrac72\tfrac32\rangle\rangle^P+|\tfrac12\tfrac32,\tfrac12\tfrac92\rangle\rangle^P+\sqrt2|\tfrac12\tfrac32,\tfrac52\tfrac12\rangle\rangle^P\Big)\Big)\,. \qquad (2.26)\end{aligned}$$

This proposal respects in addition the $\mathbb{Z}_3$-symmetry under which the three P-branes are connected: in the Landau–Ginzburg description this is the symmetry $x\to e^{\frac{2\pi i}{3}}x$, under which Q_{ξ_i} transforms into $Q_{\xi_{i-1}}$. In conformal field theory this corresponds to multiplying the contribution from a sector with the coset labels $(l_1,m_1,s_1)\otimes(l_2,m_2,s_2)$ by the phase $e^{\frac{2\pi i m_1}{3}}$.

2.2.3 The construction of the P-branes in the RR sector

The analysis in the RR sector is similar. The tensor product branes only couple to the sectors with $m_1=m_2=0$, namely to the sectors $(1,1)\otimes(1,1)$ and $(1,1)\otimes(2,2)$. In particular, the tensor product branes do not couple to any RR ground states. The generalised permutation branes on the other hand couple to all allowed RR sectors. Let us introduce the following notation for the combination of Ishibashi states that are relevant for the P-branes:

$$|1\,0,4\,0\rangle\rangle^P=|\tfrac18,0\rangle\rangle+\sum_{n=0}^{\infty}\big(|\tfrac18+3+3n,\tfrac12\rangle\rangle+|\tfrac18+3+3n,-\tfrac12\rangle\rangle\big) \qquad (2.27)$$

$$|2\,0,2\,0\rangle\rangle^P=|\tfrac18,0\rangle\rangle+\sum_{n=0}^{\infty}\big(|\tfrac18+3+3n,\tfrac12\rangle\rangle+|\tfrac18+3+3n,-\tfrac12\rangle\rangle\big) \qquad (2.28)$$

$$|2\,0,3\,1\rangle\rangle^P=\sum_{n=0}^{\infty}\big(|\tfrac58+1+3n,\tfrac12\rangle\rangle+|\tfrac58+1+3n,-\tfrac12\rangle\rangle\big) \qquad (2.29)$$

$$|1\,0,1\,3\rangle\rangle^P=\sum_{n=0}^{\infty}\big(|\tfrac58+1+3n,\tfrac12\rangle\rangle+|\tfrac58+1+3n,-\tfrac12\rangle\rangle\big) \qquad (2.30)$$

$$|1\,1,2\,2\rangle\rangle^P = \sum_{n=0}^{\infty} \big(-|\tfrac{1}{8}{+}1{+}3n,\tfrac{1}{2}\rangle\rangle + |\tfrac{1}{8}{+}1{+}3n,{-}\tfrac{1}{2}\rangle\rangle + |\tfrac{1}{8}{+}2{+}3n,\tfrac{1}{2}\rangle\rangle - |\tfrac{1}{8}{+}2{+}3n,{-}\tfrac{1}{2}\rangle\rangle \big) \tag{2.31}$$

$$|1\,1,1\,1\rangle\rangle^P = \sum_{n=0}^{\infty} \big(|\tfrac{5}{8}{+}3n,\tfrac{1}{2}\rangle\rangle - |\tfrac{5}{8}{+}3n,{-}\tfrac{1}{2}\rangle\rangle - |\tfrac{5}{8}{+}2{+}3n,\tfrac{1}{2}\rangle\rangle + |\tfrac{5}{8}{+}2{+}3n,{-}\tfrac{1}{2}\rangle\rangle \big)\ . \tag{2.32}$$

The RR part of the generalised permutation boundary states is then

$$\begin{aligned} \|P(j)\rangle\rangle_{\mathrm{RR}} = \frac{i}{\sqrt{2}} & \big(|1\,1,1\,1\rangle\rangle^P + |1\,1,2\,2\rangle\rangle^P\big) + e^{\frac{2\pi i j}{3}} \big(|1\,0,4\,0\rangle\rangle^P + |1\,0,1\,3\rangle\rangle^P\big) \\ & + e^{-\frac{2\pi i j}{3}} \big(|2\,0,2\,0\rangle\rangle^P + |2\,0,3\,1\rangle\rangle^P\big)\ . \end{aligned} \tag{2.33}$$

Finally, the complete generalised permutation branes are then simply given by

$$\begin{aligned} \|P(j)\rangle\rangle &= \|P(j)\rangle\rangle_{\mathrm{NSNS}} + \|P(j)\rangle\rangle_{\mathrm{RR}} \\ \overline{\|P(j)\rangle\rangle} &= \|P(j)\rangle\rangle_{\mathrm{NSNS}} - \|P(j)\rangle\rangle_{\mathrm{RR}}\,, \end{aligned} \tag{2.34}$$

where we have denoted anti-branes by a bar. We claim that this ansatz correctly reproduces the topological spectrum that is predicted by the matrix factorisation calculation. This will be checked next.

2.3 Comparison of the spectra

In order to compare with the results of section 2.1.1 we need to calculate three different open string spectra, the self-overlap, the relative overlap between the different P-branes, and the overlap with the tensor product branes.

2.3.1 The self-overlap

The self-overlap of the NSNS contribution $\|P(j)\rangle\rangle_{\mathrm{NSNS}}$ is in fact the same as that of the $\|T(0,0)\rangle\rangle_{\mathrm{NSNS}}$ brane. It is easily seen to equal

$$\begin{aligned} & {}_{\mathrm{NSNS}}\langle\langle P(j)\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}} \|P(j)\rangle\rangle_{\mathrm{NSNS}} \\ & = \frac{1}{2}\big(\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\big) \\ & \quad \times \big(\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q}) + \chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q}) + \chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q}) + \chi^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q})\big)\,, \end{aligned} \tag{2.35}$$

where $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$. The spectrum factorises into the part coming from the $k_1 = 1$ factor and into the part from the $k_2 = 4$ factor. For the RR sector contribution we obtain on the other hand

$$\begin{aligned} & {}_{\mathrm{RR}}\langle\langle P(j)\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}} \|P(j)\rangle\rangle_{\mathrm{RR}} \\ & = \frac{1}{2}\Big(\chi^{k=1}_{1\,1}(q)\chi^{k=4}_{1\,1}(q) + \chi^{k=1}_{1\,1}(q)\chi^{k=4}_{2\,2}(q)\Big) \\ & \quad + \Big(\chi^{k=1}_{1\,0}(q)\chi^{k=4}_{4\,0}(q) + \chi^{k=1}_{2\,0}(q)\chi^{k=4}_{2\,0}(q) + \chi^{k=1}_{2\,0}(q)\chi^{k=4}_{3\,1}(q) \\ & \qquad + \chi^{k=1}_{1\,0}(q)\chi^{k=4}_{1\,3}(q) - \chi^{k=1}_{1\,1}(q)\chi^{k=4}_{1\,1}(q) - \chi^{k=1}_{1\,1}(q)\chi^{k=4}_{2\,2}(q)\Big) \end{aligned} \tag{2.36}$$

$$= \frac{1}{2}\Big(\tilde\chi^{k=1}_{\frac12\frac12}(\tilde q)\Big(\tilde\chi^{k=4}_{\frac12\frac12}(\tilde q)+\tilde\chi^{k=4}_{\frac12\frac92}(\tilde q)+\tilde\chi^{k=4}_{\frac32\frac72}(\tilde q)-\tilde\chi^{k=4}_{\frac52\frac52}(\tilde q)+\tilde\chi^{k=4}_{\frac72\frac32}(\tilde q)+\tilde\chi^{k=4}_{\frac92\frac12}(\tilde q)\Big)$$
$$+\tilde\chi^{k=1}_{\frac32\frac12}(\tilde q)\Big(\tilde\chi^{k=4}_{\frac12\frac12}(\tilde q)-\tilde\chi^{k=4}_{\frac12\frac92}(\tilde q)+\tilde\chi^{k=4}_{\frac32\frac72}(\tilde q)-\tilde\chi^{k=4}_{\frac52\frac52}(\tilde q)-\tilde\chi^{k=4}_{\frac72\frac32}(\tilde q)+\tilde\chi^{k=4}_{\frac92\frac12}(\tilde q)\Big)$$
$$+\tilde\chi^{k=1}_{\frac12\frac32}(\tilde q)\Big(\tilde\chi^{k=4}_{\frac12\frac12}(\tilde q)+\tilde\chi^{k=4}_{\frac12\frac92}(\tilde q)-\tilde\chi^{k=4}_{\frac32\frac72}(\tilde q)-\tilde\chi^{k=4}_{\frac52\frac52}(\tilde q)+\tilde\chi^{k=4}_{\frac72\frac32}(\tilde q)-\tilde\chi^{k=4}_{\frac92\frac12}(\tilde q)\Big)\Big)\,. \qquad (2.37)$$

Here $\tilde\chi$ denotes the character twisted by $(-1)^F$. It is interesting to note that the spectrum can be written entirely in terms of products of minimal model characters.

The spectrum between the brane with itself is then the sum of these two contributions; the spectrum between brane and anti-brane is obtained by subtracting the RR contribution from the NSNS one. In either case it is clear from the above expressions that the resulting spectrum satisfies the Cardy condition, i.e. that it consists of an integer linear combination of GSO-projected $N = 2$ representations.

To compare these spectra with the results from the matrix factorisation approach, we finally have to extract the topological states. The chiral primaries in the spectrum are the ground states of the representations $(u_1, v_1) \otimes (u_2, v_2)$ that have a $v_1 = v_2 = \frac{1}{2}$. In the present case the relevant representations are

$$(\tfrac12,\tfrac12)\otimes(\tfrac12,\tfrac12)\,,\ (\tfrac32,\tfrac12)\otimes(\tfrac12,\tfrac12)\,,\ (\tfrac12,\tfrac12)\otimes(\tfrac92,\tfrac12)\,,\ (\tfrac32,\tfrac12)\otimes(\tfrac92,\tfrac12)\,, \qquad (2.38)$$

and their U(1) charges are $q = 0, \frac{1}{3}, \frac{2}{3}, 1$, respectively. In the brane-brane case all four states survive (thus we have four 'topological bosons'), while for the brane anti-brane case all are projected away (i.e. there are no 'topological fermions'.) This is then in perfect agreement with the prediction (1) in section 2.1.1.

2.3.2 Relative overlaps between the P-branes

In the NSNS sector the relative overlap of $P(j)$ with $P(j + 1)$ is

$$\begin{aligned}
&{}_{\rm NSNS}\langle\langle P(j)\| q^{\frac12(L_0+\tilde L_0)-\frac{c}{24}}\|P(j+1)\rangle\rangle_{\rm NSNS}\\
&=\frac{\sqrt3}{2}\Big(\chi^{k=1}_{\frac12\frac12}(q)\chi^{k=4}_{\frac12\frac12}(q)-\frac43\sum_{\substack{m=1\\ m \text{ or } n \text{ odd}}}^{\infty}\sum_{n=1}^{m}\chi^{c=3,{\rm NS}}_{(m^2+n^2-mn,0)}(q)\\
&\quad+\chi^{k=1}_{\frac12\frac12}(q)\chi^{k=4}_{\frac52\frac52}(q)-\frac43\sum_{\substack{m=1\\ m \text{ or } n \text{ odd}}}^{\infty}\sum_{n=1}^{m}\chi^{c=3,{\rm NS}}_{(m^2+n^2-mn,0)}(q)\\
&\quad+2\chi^{k=1}_{\frac12\frac12}(q)\chi^{k=4}_{\frac32\frac32}(q)-\frac83\sum_{\substack{m=0\\ m+1 \text{ or } n \text{ odd}}}^{\infty}\sum_{n=0}^{\infty}\chi^{c=3,{\rm NS}}_{(\frac13+m^2+n^2-mn+n,0)}(q)\Big)\\
&={}_{\rm NSNS}\langle\langle P(j)\| q^{\frac12(L_0+\tilde L_0)-\frac{c}{24}}\|P(j)\rangle\rangle_{\rm NSNS}\ +\ \Pi(q)\,,
\end{aligned} \qquad (2.39)$$

where Π denotes the difference to the self-overlap (2.35) of the $P(j)$-brane. It is given by

$$\begin{aligned}\Pi(q) &= -\frac{4}{\sqrt{3}}\Big(\sum_{\substack{m=1\\ m \text{ or } n \text{ odd}}}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{(m^2+n^2-mn,0)}(q)+\sum_{\substack{m=0\\ m+1 \text{ or } n \text{ odd}}}^{\infty}\sum_{n=0}^{\infty}\chi^{c=3,\mathrm{NS}}_{(\frac{1}{3}+m^2+n^2-mn+n,0)}(q)\Big)\\ &= -\frac{4}{\sqrt{3}}\sum_{\substack{m=1\\ m \text{ or } n \text{ odd}}}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{(\frac{m^2+n^2-mn}{3},0)}(q)\\ &= -\frac{2}{3\sqrt{3}}\Big(\sum_{m,n\in\mathbb{Z}}q^{\frac{m^2+n^2-mn}{3}}-\sum_{\substack{m,n\in\mathbb{Z}\\ m,n \text{ even}}}q^{\frac{m^2+n^2-mn}{3}}\Big)\frac{\vartheta_3(q)}{\eta^3(q)}\ . \end{aligned} \tag{2.40}$$

Modular transformation of this term leads to

$$\begin{aligned}\tilde{\Pi}(\tilde{q}) &= -\frac{2}{3\sqrt{3}}\Big(\sqrt{3}\sum_{m,n\in\mathbb{Z}}\tilde{q}^{m^2+n^2-mn}-\frac{\sqrt{3}}{4}\sum_{m,n\in\mathbb{Z}}\tilde{q}^{\frac{m^2+n^2-mn}{4}}\Big)\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})}\\ &= -4\Big(\sum_{m=1}^{\infty}\sum_{n=1}^{m}\tilde{q}^{m^2+n^2-mn}-\frac{1}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{m}\tilde{q}^{\frac{m^2+n^2-mn}{4}}\Big)\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})}-\frac{1}{2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})}\\ &= -3\sum_{m=1}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{(m^2+n^2-mn,0)}(\tilde{q})+\sum_{\substack{m=1\\ m \text{ or } n \text{ odd}}}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{(\frac{m^2+n^2-mn}{4},0)}(\tilde{q})-\frac{1}{2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})}\\ &= \sum_{\substack{m=1\\ m \text{ or } n \text{ odd}}}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{(\frac{m^2+n^2-mn}{4},0)}(\tilde{q})-\frac{1}{2}\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})\Big(\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})+\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q})\Big)\\ &\quad -\frac{1}{2}\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q})\Big(\chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})+\chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q})\Big)\\ &\quad -\frac{1}{2}\chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\Big(\chi^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q})+\chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q})\Big)\ , \end{aligned} \tag{2.41}$$

where in the last step we used (B.29). Inserting this and (2.35) in (2.39) we finally obtain

$$\begin{aligned}&{}_{\mathrm{NSNS}}\langle\langle P(j)\|q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}}\|P(j+1)\rangle\rangle_{\mathrm{NSNS}}\\ &\quad =\frac{1}{2}\bigg(\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})\Big(\chi^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q})+\chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q})+\chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q})+\chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})\Big)\\ &\qquad +\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q})\Big(\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})+\chi^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q})+\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q})+\chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q})\Big)\\ &\qquad +\chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\Big(\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})+\chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q})+\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q})+\chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})\Big)\bigg)\\ &\quad +\sum_{\substack{m=1\\ m \text{ or } n \text{ odd}}}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{(\frac{m^2+n^2-mn}{4},0)}(\tilde{q})\ . \end{aligned} \tag{2.42}$$

Note that the spectrum can be described as the spectral flow orbits of the two chiral primaries $(\frac{1}{2},\frac{1}{2})\otimes(\frac{9}{2},\frac{1}{2})$ and $(\frac{3}{1},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2})$, and the tower of representations with charge zero for which each representation is spectral flow invariant.

The overlap between different permutation branes in the RR sector is

$$
\begin{aligned}
&{}_{\mathrm{RR}}\langle\langle P(j)\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}} \| P(j+1)\rangle\rangle_{\mathrm{RR}} \\
&= \frac{1}{2}\Big(\chi^{k=1}_{1\,1}(q)\chi^{k=4}_{1\,1}(q) + \chi^{k=1}_{1\,1}(q)\chi^{k=4}_{2\,2}(q)\Big) \\
&\quad - \frac{1}{2}\Big(\chi^{k=1}_{1\,0}(q)\chi^{k=4}_{4\,0}(q) + \chi^{k=1}_{2\,0}(q)\chi^{k=4}_{2\,0}(q) + \chi^{k=1}_{2\,0}(q)\chi^{k=4}_{3\,1}(q) \\
&\qquad + \chi^{k=1}_{1\,0}(q)\chi^{k=4}_{1\,3}(q) - \chi^{k=1}_{1\,1}(q)\chi^{k=4}_{1\,1}(q) - \chi^{k=1}_{1\,1}(q)\chi^{k=4}_{2\,2}(q)\Big) \qquad (2.43) \\
&= -\frac{1}{2}\Big(\tilde{\chi}^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})\Big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q}) + \tilde{\chi}^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q}) + \tilde{\chi}^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q}) + \tilde{\chi}^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})\Big) \\
&\qquad + \tilde{\chi}^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q})\Big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) - \tilde{\chi}^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q}) - \tilde{\chi}^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q}) - \tilde{\chi}^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q})\Big) \\
&\qquad + \tilde{\chi}^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\Big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) - \tilde{\chi}^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q}) - \tilde{\chi}^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q}) - \tilde{\chi}^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})\Big)\Big)\,. \qquad (2.44)
\end{aligned}
$$

Combining these two expressions we thus see that the sum (i.e. the overlap between brane and brane) does not contain any chiral primaries. On the other hand the relative spectrum of $P(j)$ and $\overline{P(j+1)}$ contains one chiral primary in the sector $(\frac{1}{2},\frac{1}{2})\otimes(\frac{9}{2},\frac{1}{2})$ with charge $q=\frac{2}{3}$ and a second in the sector $(\frac{3}{2},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2})$ with charge $q=\frac{1}{3}$. This coincides with the topological spectrum in prediction (2) in section 2.1.1.

2.3.3 The relative overlap between P-branes and the $T(0,0)$-brane

Finally, we consider the relative overlap between $P(j)$ and $T(0,0)$. It is very easy to see that it is independent of j. In the NSNS sector the result is actually the same as for the relative overlap between $P(j)$ and $P(j+1)$ (2.42). In the RR sector we find on the other hand

$$
\begin{aligned}
&{}_{\mathrm{RR}}\langle\langle T(0,0)\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}} \| P(j)\rangle\rangle_{\mathrm{RR}} \\
&= -\frac{i\sqrt{3}}{2}\sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{1}{8}+1+3n,\frac{1}{2})}(q) - \chi^{c=3,\mathrm{NS}}_{(\frac{1}{8}+1+3n,-\frac{1}{2})}(q) - \chi^{c=3,\mathrm{NS}}_{(\frac{1}{8}+2+3n,\frac{1}{2})}(q) + \chi^{c=3,\mathrm{NS}}_{(\frac{1}{8}+2+3n,-\frac{1}{2})}(q) \\
&\qquad - \chi^{c=3,\mathrm{NS}}_{(\frac{5}{8}+3n,\frac{1}{2})}(q) + \chi^{c=3,\mathrm{NS}}_{(\frac{5}{8}+3n,-\frac{1}{2})}(q) + \chi^{c=3,\mathrm{NS}}_{(\frac{5}{8}+2+3n,\frac{1}{2})}(q) - \chi^{c=3,\mathrm{NS}}_{(\frac{5}{8}+2+3n,-\frac{1}{2})}(q)\Big) \qquad (2.45) \\
&= -\frac{i\sqrt{3}}{2}\Big(- \chi^{k=1}_{2\,0}(q)\chi^{k=4}_{2\,0}(q) + \chi^{k=1}_{1\,0}(q)\chi^{k=4}_{4\,0}(q) \\
&\qquad - \chi^{k=1}_{2\,0}(q)\chi^{k=4}_{3\,1}(q) + \chi^{k=1}_{1\,0}(q)\chi^{k=4}_{1\,3}(q)\Big) \qquad (2.46) \\
&= +\frac{1}{2}\tilde{\chi}^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\Big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) + \tilde{\chi}^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q}) - \tilde{\chi}^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q}) + \tilde{\chi}^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q})\Big) \\
&\quad - \frac{1}{2}\tilde{\chi}^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q})\Big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) + \tilde{\chi}^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q}) - \tilde{\chi}^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q}) + \tilde{\chi}^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q})\Big) \\
&\quad - \frac{1}{2}\tilde{\chi}^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})\Big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q}) + \tilde{\chi}^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q}) - \tilde{\chi}^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q}) - \tilde{\chi}^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})\Big)\,. \qquad (2.47)
\end{aligned}
$$

Here we used the character decompositions of appendix B.2 in the first step. By adding the two contributions (eqs. (2.42) and (2.47)) we thus find that the relative spectrum between $T(0,0)$ and $P(j)$ contains a chiral primary in the sector $(\frac{1}{2},\frac{1}{2})\otimes(\frac{9}{2},\frac{1}{2})$ of charge $q=\frac{2}{3}$.

The relative spectrum between $T(0,0)$ and $\overline{P(j)}$ on the other hand contains one chiral primary in the sector $(\frac{3}{2},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2})$ of charge $q=\frac{1}{3}$. This is then again consistent with the prediction (3) of section 2.1.1.

The correct topological spectrum is a strong indication that the branes we have constructed above correspond indeed to the generalised permutation factorisations. We shall now give further evidence for the consistency of the P-branes by relating them to constructions in a free orbifold theory.

3. The torus orbifold

The $A_1 \times A_4$ we discussed in the previous section has $c = 3$, and one may therefore expect that it is in fact equivalent to a free field theory. This is indeed the case: the theory is equivalent to a $\mathbb{Z}_6$-orbifold of the $su(3)$ torus [23]. The boundary states we have constructed above should thus have a description in terms of this free field theory; this will now be explained.

3.1 The orbifold theory

We consider the $N = 2$ theory of two bosons and two fermions on the $su(3)$ torus. The torus is described by the two-dimensional plane with identifications given by two vectors, e_1, e_2 where $e_1\cdot e_1 = e_2\cdot e_2 = 2$ and $e_1\cdot e_2 = -1$. If we think of the plane as $\mathbb{C}$, then we can choose e_1, e_2 as two vectors of equal length with e_1 pointing along the positive real axis, and e_2 pointing in the direction of $e^{\frac{2\pi i}{3}}$. The winding on the torus is then described by the lattice Λ_R spanned by e_1 and e_2. The momentum takes values in the dual lattice Λ_W spanned by

$$e_1^* = \frac{2}{3}e_1 + \frac{1}{3}e_2\,, \qquad e_2^* = \frac{1}{3}e_1 + \frac{2}{3}e_2 \tag{3.1}$$

with the inner products $e_1^*\cdot e_1^* = e_2^*\cdot e_2^* = \frac{2}{3}$ and $e_1^*\cdot e_2^* = \frac{1}{3}$. The lattices Λ_R and Λ_W are the root and weight lattice of $su(3)$, respectively.
The theory that is of interest to us has in addition a B-field, and the left- and right-moving momenta are thus given by

$$p_L = p + \tfrac{1}{2}L - \tfrac{1}{2}BL\,, \qquad p_R = p - \tfrac{1}{2}L - \tfrac{1}{2}BL\,, \tag{3.2}$$

where $p \in \Lambda_W$ and $L \in \Lambda_R$. The B-field is a matrix chosen such that $Be_1 = -e_2^*$ and $Be_2 = e_1^*$. Then the lattice $\Gamma_{2,2}$ of the momenta (p_L, p_R) is

$$\Gamma_{2,2} = \{(p_L, p_R) \in \Lambda_W \oplus \Lambda_W\,,\ p_L - p_R \in \Lambda_R\}\ . \tag{3.3}$$

The resulting partition function is (we are considering a type 0B like GSO-projection)

$$Z_{\text{Torus}} = \frac{1}{|\eta(q)|^4} \sum_{(p_L,p_R)\in\Gamma_{2,2}} q^{\frac{1}{2}(p_L)^2}\bar{q}^{\frac{1}{2}(p_R)^2} \left(\frac{|\vartheta_3(q)|^2}{|\eta(q)|^2} + \frac{|\vartheta_2(q)|^2}{|\eta(q)|^2} + \frac{|\vartheta_4(q)|^2}{|\eta(q)|^2}\right)\,. \tag{3.4}$$

Our conventions for the ϑ and η functions are summarised in appendix B.4. In the following we shall use the left-moving modes α_n and $\bar{\alpha}_n$ corresponding to complex target space coordinates; the right-moving modes will be denoted by a tilde. Similarly, we use complex fermionic oscillator modes ψ_r^+ and ψ_r^-. The commutation relations are then

$$[\alpha_m, \bar{\alpha}_n] = m\, \delta_{m,-n}\,, \qquad \{\psi_r^+, \psi_s^-\} = \delta_{r,-s}\,, \tag{3.5}$$

with similar relations for the right-movers. The $\mathbb{Z}_6$-orbifold of interest acts on the complex coordinates as $z \to e^{2\pi i/6} z$. It is not difficult to show that the spectrum of this orbifold theory agrees indeed with (2.2).

3.2 Branes on the $su(3)$ torus

The branes we are interested in are D1-branes on the torus with Wilson lines.[3] The relevant gluing conditions are

$$(\alpha_n + e^{2i\phi}\tilde{\bar{\alpha}}_{-n})\,\|B\rangle\!\rangle = 0 \qquad (\bar{\alpha}_n + e^{-2i\phi}\tilde{\alpha}_{-n})\,\|B\rangle\!\rangle = 0 \tag{3.6}$$

$$(\psi_r^+ - ie^{2i\phi}\eta\tilde{\psi}_{-r}^-)\,\|B\rangle\!\rangle = 0 \qquad (\psi_r^- + ie^{-2i\phi}\eta\tilde{\psi}_{-r}^+)\,\|B\rangle\!\rangle = 0\,. \tag{3.7}$$

The boundary states are linear combinations of the coherent states

$$|\phi, p_L, \eta\rangle\!\rangle = \exp\Big(-\sum_{n=1}^{\infty} \frac{1}{n}\big(e^{-2i\phi}\alpha_{-n}\tilde{\alpha}_{-n} + e^{2i\phi}\bar{\alpha}_{-n}\tilde{\bar{\alpha}}_{-n}\big) \tag{3.8}$$

$$- i\eta \sum_{r>0}\big(e^{-2i\phi}\psi_{-r}^+\tilde{\psi}_{-r}^+ - e^{2i\phi}\psi_{-r}^-\tilde{\psi}_{-r}^-\big)\Big)|p_L, p_R = -e^{2i\phi}\bar{p}_L\rangle\,, \tag{3.9}$$

where $|p_L, p_R\rangle$ denotes the ground state with the corresponding momenta. Note that the GSO projection (type 0B) is chosen such that the boundary states of the torus theory (before orbifolding) will not have any RR contribution. As before, we only want to consider branes with a fixed spin-structure η, say $\eta = +1$.
Before orbifolding, the full boundary state is then

$$\|\tilde{B}(u,\phi)\rangle\!\rangle \;=\; \mathcal{N}_\phi \sum_{p_L} e^{2\pi i (u, p_L)}|\phi, p_L\rangle\!\rangle\,. \tag{3.10}$$

Here u is a vector which contains information on the position and the Wilson line on the brane. The sum over p_L is restricted to those p_L for which p_L and $p_R = -e^{2i\phi}\bar{p}_L$ are allowed combinations of momenta. In particular, because of (3.2), the winding L has to be in the direction given by $e^{i\phi}$ (the direction of the brane), and the momentum p is perpendicular to the brane. The phase in front of the Ishibashi states in (3.10) can be written as

$$e^{2\pi i (u, p_L)} = e^{2\pi i [u_\perp p + \frac{1}{2}(u+Bu)_\parallel L]}\,, \tag{3.11}$$

where $u_\parallel$ denotes the projection of u in the direction of the brane, and $u_\perp$ is the orthogonal complement. The boundary state (3.10) thus corresponds to a D1-brane in a direction given by $e^{i\phi}$ in the complex plane with a position given by $u_\perp$ and a Wilson line determined by $\frac{1}{2}(u+Bu)_\parallel$. Note that the value of u is a priori only defined modulo an overall shift which corresponds to a redefinition of the Ishibashi states by a phase.

[3] This is similar to the situation in [29], see also [30].

There are two types of angles that will be relevant in the following. If $\phi_n = \frac{2\pi n}{3}$, then all momenta $p \in \Lambda_W$ are allowed in the sum in (3.10). In this case the correctly normalised boundary state is

$$\|\tilde{B}(u,\phi_n)\rangle\rangle \;=\; 3^{-\frac{1}{4}} \sum_{p_L\in\Lambda_W} e^{2\pi i(u,p_L)}|\phi_n,p_L\rangle\rangle \;. \tag{3.12}$$

Its self-spectrum is

$$\langle\langle\tilde{B}(u,\phi_n)\|q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}}\|\tilde{B}(u,\phi_n)\rangle\rangle = \frac{1}{\sqrt{3}}\sum_{p\in\Lambda_W} q^{\frac{1}{2}p^2}\frac{\vartheta_3(q)}{\eta^3(q)} = \sum_{p\in\Lambda_R}\tilde{q}^{\frac{1}{2}p^2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})} \;. \tag{3.13}$$

Here $\vartheta_3(q)$ denotes a Jacobi theta function, and $\eta(q)$ is the Dedekind function. We also note that the relative spectrum between two branes $\|\tilde{B}(u,\phi)\rangle\rangle$ with $\phi_n = \frac{2\pi n}{3}$ but different labels u,u' is

$$\langle\langle\tilde{B}(u',\phi_n)\|q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}}\|\tilde{B}(u,\phi_n)\rangle\rangle = \frac{1}{\sqrt{3}}\sum_{p\in\Lambda_W} q^{\frac{1}{2}p^2}e^{2\pi i(u-u',p)}\frac{\vartheta_3(q)}{\eta^3(q)} \tag{3.14}$$

$$= \sum_{p\in\Lambda_R}\tilde{q}^{\frac{1}{2}(p+u-u')^2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})} \;. \tag{3.15}$$

Finally, the relative spectrum between branes at different angles ϕ_n and $\phi_{n'}$ is $(n \neq n')$

$$\langle\langle\tilde{B}(u',\phi_{n'})\|q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}}\|\tilde{B}(u,\phi_n)\rangle\rangle = \frac{\vartheta_3(e^{2\pi i/3},q)}{\vartheta_1(e^{2\pi i/3},q)} = -i\frac{\vartheta_3(\tilde{q}^{-1/3},\tilde{q})}{\vartheta_1(\tilde{q}^{-1/3},\tilde{q})} \;. \tag{3.16}$$

The other case of interest is $\phi = \psi_n = 2\pi(\frac{1}{12}+\frac{n}{6})$ with $n = 0,1,2$. The corresponding boundary states then only couple to momenta (p_L,p_R) from the root lattice, and the correctly normalised boundary state is

$$\|\tilde{B}(u,\psi_n)\rangle\rangle \;=\; 3^{\frac{1}{4}} \sum_{p_L\in\Lambda_R} e^{2\pi i(u,p_L)}|\psi_n,p_L\rangle\rangle \tag{3.17}$$

with the self-spectrum

$$\langle\langle\tilde{B}(u,\psi_n)\|q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}}\|\tilde{B}(u,\psi_n)\rangle\rangle = \sum_{p\in\Lambda_W}\tilde{q}^{\frac{1}{2}p^2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})} \;. \tag{3.18}$$

3.3 Branes on the orbifold

It is convenient to perform the $\mathbb{Z}_6$-orbifold operation in two steps, first by performing the $\mathbb{Z}_2$-part (reflection). In a second step we then perform the remaining $\mathbb{Z}_3$-orbifold (rotation by 120 degrees). In the first step there are the four $\mathbb{Z}_2$-fixed points (see figure 1). For each of the four fixed points we have a $\mathbb{Z}_2$-twisted sector. The second $\mathbb{Z}_3$-operation leaves the fixed point (0) invariant, and permutes the three fixed points (1) $\mapsto$ (2) $\mapsto$ (3). In addition to (0) it has two further fixed points that lead again to twisted sectors. Some of the branes are invariant under the reflection, and thus couple to $\mathbb{Z}_2$-twisted sectors; however, none will be fixed under the second $\mathbb{Z}_3$-orbifold, and we will therefore not have to resolve any fixed points in the second step.

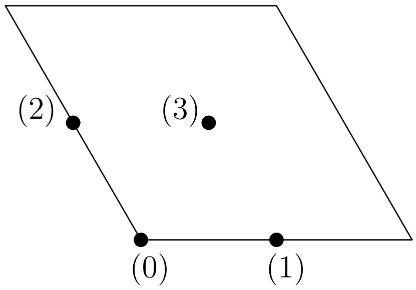

Figure 1: Fixed points

Without loss of generality we may choose the phase factor of the above Ishibashi states so that they transform trivially under a a rotation by $e^{2\pi i/6}$

$$|\phi, p\rangle\rangle \to |\phi + \tfrac{2\pi}{6}, e^{2\pi i/6} p\rangle\rangle \ . \tag{3.19}$$

3.3.1 Orbifolding by the $\mathbb{Z}_2$-subgroup

The D1-brane is fixed under the $\mathbb{Z}_2$-action if u is of the form (written in the basis (e_1, e_2))

$$u_0 = (0,0) \qquad u_1 = (\tfrac{1}{2}, 0) \tag{3.20}$$
$$u_2 = (0, \tfrac{1}{2}) \qquad u_3 = (\tfrac{1}{2}, \tfrac{1}{2}) \ . \tag{3.21}$$

If u is of this form, the corresponding brane couples to some linear combination of the four $\mathbb{Z}_2$-twisted sectors. With our choice of GSO projection we do not get any Ishibashi states from the twisted NSNS sector, but we do obtain twisted RR contributions which we denote by $|(i), \phi\rangle\rangle$; we chose the convention that Ishibashi states corresponding to different angles ϕ have the same phase in front of the ground state. Hence their overlaps are

$$\langle\langle (i), \phi' | q^{\frac{1}{2}(L_0 + \tilde{L}_0) - \frac{c}{24}} z^{\frac{1}{2}(J_0 + \tilde{J}_0)} | (i), \phi\rangle\rangle = \frac{\vartheta_3(z e^{2i(\phi' - \phi)}, q)}{\vartheta_4(e^{2i(\phi'-\phi)}, q)} = e^{-\frac{i\pi w^2}{\tau}} \tilde{z}^{\frac{\phi - \phi'}{\pi}} \frac{\vartheta_3(\tilde{z} \tilde{q}^{\frac{\phi-\phi'}{\pi}}, \tilde{q})}{\vartheta_2(\tilde{q}^{\frac{\phi-\phi'}{\pi}}, \tilde{q})} \ . \tag{3.22}$$

We included the U(1) charges which we shall need later, and introduced the notation $z = e^{2\pi i w}$ and $\tilde{z} = e^{2\pi i w/\tau}$.

Depending on the Wilson line and the orientation, the different D1-branes couple differently to the twisted sectors. For $\phi = 0$, we have formally the structure ((i) denotes the contribution of the i^{th} twisted sector)

$$u_0 : (0) + (1) \qquad u_1 : (1) - (0) \tag{3.23a}$$
$$u_2 : \ -(2) - (3) \qquad u_3 : (2) - (3) \ . \tag{3.23b}$$

Analogously we then have for $\phi = 2\pi/3$:

$$u_0 : (0) + (2) \qquad u_2 : (2) - (0) \tag{3.23c}$$
$$u_3 : \ -(3) - (1) \qquad u_1 : (3) - (1) \, , \tag{3.23d}$$

and for $\phi = 4\pi/3$:

$$u_0 : (0) + (3) \qquad u_3 : (3) - (0) \tag{3.23e}$$
$$u_1 : \ -(1) - (2) \qquad u_2 : (1) - (2) \ . \tag{3.23f}$$

A boundary state in the $\mathbb{Z}_2$-orbifold is then

$$\| B(u_i, \phi_n)\rangle\rangle = \frac{1}{\sqrt{2}} \| \tilde{B}(u_i, \phi_n)\rangle\rangle + \frac{1}{\sqrt{2}} \big(\text{twisted sector contribution}\big) \ . \tag{3.24}$$

The corresponding anti-brane has the opposite sign in front of the twisted sector contribution. The spectrum of the resolved branes is then given by

$$\langle\langle B(u_i,\phi_n)\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}} \| B(u_i,\phi_n)\rangle\rangle = \frac{1}{2}\sum_{p\in\Lambda_R}\tilde{q}^{\frac{1}{2}p^2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})}+\frac{\vartheta_3(\tilde{q})}{\vartheta_2(\tilde{q})}\ . \tag{3.25}$$

With these preparations we can now specify all the relative spectra between the resolved branes. If the angles are equal, but the Wilson line parameters $u_i, u_{i'}$ are different, then there is no contribution from the twisted sector and we find

$$\langle\langle B(u_i',\phi_n)\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}} \| B(u_i,\phi_n)\rangle\rangle = \frac{1}{2}\sum_{p\in\Lambda_R}\tilde{q}^{\frac{1}{2}(p+u_1)^2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})}\ . \tag{3.26}$$

Here we have used that $u_{i'} - u_i = u_1 \mod \Lambda_R$ for $i \neq i'$.

If the angles are different, there is always a contribution from the twisted sector, and the spectrum is given by

$$\frac{1}{2}\Big(-i\frac{\vartheta_3(\tilde{q}^{-1/3},\tilde{q})}{\vartheta_1(\tilde{q}^{-1/3},\tilde{q})}\pm\frac{\vartheta_3(\tilde{q}^{-1/3},\tilde{q})}{\vartheta_2(\tilde{q}^{-1/3},\tilde{q})}\Big)\ . \tag{3.27}$$

The sign depends on the couplings to the twisted sectors; this can be determined from (3.23).

3.3.2 The final $\mathbb{Z}_3$-orbifold

We can now take superpositions of these D-branes under the remaining $\mathbb{Z}_3$-action to turn them into branes of the $\mathbb{Z}_6$-orbifold. In particular, we find the four branes that we label as

$$\|T(0,0)\rangle\rangle = \frac{1}{\sqrt{3}}\big(\|B(u_3,\phi=0)\rangle\rangle + \|B(u_1,\phi=\tfrac{2\pi}{3})\rangle\rangle + \|B(u_2,\phi=\tfrac{4\pi}{3})\rangle\rangle\big) \tag{3.28}$$

$$\|P(1)\rangle\rangle = \frac{1}{\sqrt{3}}\big(\|B(u_0,\phi=0)\rangle\rangle + \|B(u_0,\phi=\tfrac{2\pi}{3})\rangle\rangle + \|B(u_0,\phi=\tfrac{4\pi}{3})\rangle\rangle\big) \tag{3.29}$$

$$\|P(2)\rangle\rangle = \frac{1}{\sqrt{3}}\big(\|B(u_1,\phi=0)\rangle\rangle + \|B(u_2,\phi=\tfrac{2\pi}{3})\rangle\rangle + \|B(u_3,\phi=\tfrac{4\pi}{3})\rangle\rangle\big) \tag{3.30}$$

$$\|P(3)\rangle\rangle = \frac{1}{\sqrt{3}}\big(\|B(u_2,\phi=0)\rangle\rangle + \|B(u_3,\phi=\tfrac{2\pi}{3})\rangle\rangle + \|B(u_1,\phi=\tfrac{4\pi}{3})\rangle\rangle\big)\ . \tag{3.31}$$

The corresponding anti-branes differ in their coupling to the twisted RR sectors. Figures 2 and 3 show an illustration of these branes on the torus. Note that the branes $P(1)$ and $P(2)$ ($T(0,0)$ and $P(3)$) are at the same location, but carry different Wilson lines which is not visible in the figure.

We now claim that these branes agree indeed with the correspondingly labelled branes of the $N = 2$ theory. In the following we shall confirm this by determining the relative spectra and comparing them with the results of the previous section.

3.4 Comparison of the overlaps

It is straightforward to determine the self-spectra of these orbifold branes. They are

$$\langle\langle T(0,0)\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}} \| T(0,0)\rangle\rangle = \frac{1}{2}\sum_{p\in\Lambda_R}\tilde{q}^{\frac{1}{2}p^2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})}+\frac{\vartheta_3(\tilde{q})}{\vartheta_2(\tilde{q})}$$

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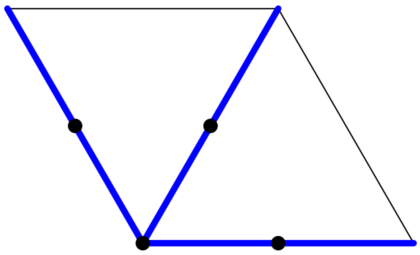

Figure 2: The superposition of branes that make up $P(1)$ or $P(2)$.

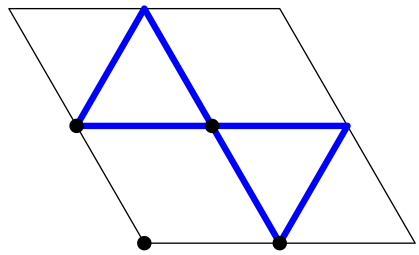

Figure 3: The superposition of branes that make up $T(0,0)$ or $P(3)$.

$$+\left(-i\frac{\vartheta_3(\tilde{q}^{-1/3},\tilde{q})}{\vartheta_1(\tilde{q}^{-1/3},\tilde{q})}-\frac{\vartheta_3(\tilde{q}^{-1/3},\tilde{q})}{\vartheta_2(\tilde{q}^{-1/3},\tilde{q})}\right) \tag{3.32}$$

and

$$\begin{aligned}\langle\langle P(j)\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}}\|P(j)\rangle\rangle =&\frac{1}{2}\sum_{p\in\Lambda_R}\tilde{q}^{\frac{1}{2}p^2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})}+\frac{\vartheta_3(\tilde{q})}{\vartheta_2(\tilde{q})}\\ &+\left(-i\frac{\vartheta_3(\tilde{q}^{-1/3},\tilde{q})}{\vartheta_1(\tilde{q}^{-1/3},\tilde{q})}+\frac{\vartheta_3(\tilde{q}^{-1/3},\tilde{q})}{\vartheta_2(\tilde{q}^{-1/3},\tilde{q})}\right).\end{aligned} \tag{3.33}$$

The only difference in the spectra of the $T(0,0)$-brane and the $P(j)$-branes is the sign in the second line.
The relative spectrum between the different $P(j)$-branes is given by

$$\begin{aligned}\langle\langle P(j)\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}}\|P(j+1)\rangle\rangle =&\frac{1}{2}\sum_{p\in\Lambda_R}\tilde{q}^{\frac{1}{2}(p+u_1)^2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})}\\ &+\left(-i\frac{\vartheta_3(\tilde{q}^{-1/3},\tilde{q})}{\vartheta_1(\tilde{q}^{-1/3},\tilde{q})}-\frac{\vartheta_3(\tilde{q}^{-1/3},\tilde{q})}{\vartheta_2(\tilde{q}^{-1/3},\tilde{q})}\right).\end{aligned} \tag{3.34}$$

For the relative spectrum between $T(0,0)$ and $P(j)$ we should keep track of the $U(1)$-charges in the spectrum since otherwise the RR contributions cancel in the overlap. We obtain

$$\begin{aligned}&\langle\langle T(0,0)\| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}}z^{\frac{1}{2}(J_0+\tilde{J}_0)}\|P(j)\rangle\rangle\ e^{\frac{i\pi w^2}{\tau}}\\ &\quad=\frac{1}{2}\sum_{p\in\Lambda_R}\tilde{q}^{\frac{1}{2}(p+u_1)^2}\frac{\vartheta_3(\tilde{z},\tilde{q})}{\eta^3(\tilde{q})}\\ &\quad\quad+\frac{1}{2}\left(-i\tilde{z}^{-\frac{1}{3}}\frac{\vartheta_3(\tilde{z}\tilde{q}^{-1/3},\tilde{q})}{\vartheta_1(\tilde{q}^{-1/3},\tilde{q})}-i\tilde{z}^{\frac{1}{3}}\frac{\vartheta_3(\tilde{z}^{-1}\tilde{q}^{-1/3},\tilde{q})}{\vartheta_1(\tilde{q}^{-1/3},\tilde{q})}\right.\\ &\quad\quad\quad\left.+\tilde{z}^{-\frac{1}{3}}\frac{\vartheta_3(\tilde{z}\tilde{q}^{-1/3},\tilde{q})}{\vartheta_2(\tilde{q}^{-1/3},\tilde{q})}-\tilde{z}^{\frac{1}{3}}\frac{\vartheta_3(\tilde{z}^{-1}\tilde{q}^{-1/3},\tilde{q})}{\vartheta_2(\tilde{q}^{-1/3},\tilde{q})}\right).\end{aligned} \tag{3.35}$$

Using the formulae of appendix B.4, these spectra are easily seen to reproduce those of section 2.3.

Actually, one can also identify the other two tensor product branes ($T(0,1)$ and $T(0,2)$) in the orbifold theory. The $T(0,1)$ brane corresponds to a superposition of branes at angles

ψ_n, perpendicular to the branes at angles ϕ_n which made up the $T(0,0)$-brane. To see this one observes that the relative angle to the $T(0,0)$-brane can be read off from the phases in front of the Ishibashi states which come from the vacuum sector of the torus. The $T(0,1)$-brane couples with a relative sign to the sector $(\frac{1}{2},\frac{1}{2})\otimes(\frac{5}{2},\frac{5}{2})$, and thus in the relative overlap of the $T(0,0)$- and the $T(0,1)$-brane the states with total conformal weight $H=1$ (and total charge $Q=0$) do not contribute. This means that the $T(0,1)$-brane is perpendicular to the $T(0,0)$ brane.

By the same token it follows that the $T(0,2)$-brane is again a superposition of branes at angles ϕ_n. Its parameter u cannot be one of the four fixed values u_i as it does not couple to the twisted sector. A detailed analysis shows that for the $\phi=0$ direction the parameter u is given by $u=u_3+(\frac{1}{3},\frac{2}{3})$.

We have thus identified the various branes, including in particular the generalised permutation branes $P(j)$, with standard constructions of the $T^2/\mathbb{Z}_6$ orbifold. This gives strong support to the claim that these $P(j)$ branes are indeed consistent.

3.5 The factorisation constraint

Another consistency condition that can at least partially be checked for these D-branes is the sewing constraint of [31] that is sometimes referred to as the factorisation constraint [32] or the 'classifying algebra' [33]. It requires that the coefficients in front of the Ishibashi states of every consistent (fundamental) boundary state satisfy a quadratic equation. (This equation comes from considering different limits of the 2-point function of two bulk fields in the presence of the boundary — see for example [34].) In general this quadratic equation is difficult to determine since it requires knowledge of the actual operator product expansion coefficients of the bulk theory, as well as the fusing matrices. However, in our situation, at least some of these relations can be easily found.

Consider for example the (uncharged) primary fields of the diagonal $N=2$ algebra that originate from the tensor product of the two NS vacuum representations. As follows from the analysis of appendix B, in particular (B.1), these primary fields are labelled by (m,n) where $m\geq n\geq 1$, as well as the vacuum $(m,n)=(0,0)$. Since these fields arise in the 'vacuum sector' their operator product expansion must close among themselves, and they must therefore give rise to a factorisation constraint. In fact, in terms of the $\mathbb{Z}_6$-orbifold description, the primary field (m,n) corresponds to the $\mathbb{Z}_6$-orbit that contains the momentum state corresponding to $p=me_1+ne_2$. Given that the operator product expansion and the fusing matrices of pure momentum states are very simple, it then follows that the corresponding factorisation constraint is

$$\begin{aligned}B_{m_1,n_1}B_{m_2,n_2}=\frac{1}{6}\Big(&B_{m_1+m_2,n_1+n_2}+B_{m_1-n_2,n_1+m_2-n_2}+B_{m_1-m_2+n_2,n_1-m_2}\\&+B_{m_1-m_2,n_1-n_2}+B_{m_1+n_2,n_1-m_2+n_2}+B_{m_1+m_2-n_2,n_1+m_2}\Big)\,.\end{aligned}\tag{3.36}$$

Here $B_{m,n}$ is the coefficient of the (m,n) Ishibashi state from the $(\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2})$ sector (normalised so that $B_{0,0}=1$). Furthermore, it is understood that we have the identifications that come from the $\mathbb{Z}_6$-orbit

$$B_{m,n}=B_{-n,m-n}=B_{n-m,-m}=B_{-m,-n}=B_{n,n-m}=B_{m-n,m}\;;\tag{3.37}$$

this allows us to rewrite every term on the right-hand-side in terms of a coefficient corresponding to (m,n) with $m \geq n \geq 1$ or $(m,n) = (0,0)$.

It is obvious that the original tensor branes (for which $B_{m,n} = 1$ for all (m,n)) satisfy (3.36). It is also easy to check (but a nontrivial consistency check!) that the same is the case for our P-branes, for which we have (see (2.17))

$$B^P_{m,n} = \begin{cases} 1 & \text{if } m \text{ and } n \text{ are even} \\ -\frac{1}{3} & \text{otherwise.} \end{cases} \tag{3.38}$$

The fact that many of these coefficients $B_{m,n}$ are equal to 1 suggests that the boundary state preserves in fact a large W-algebra. This W-algebra is generated by all the $N = 2$ primary fields from the tensor product of the two vacuum representations corresponding to (m,n) with both m and n even.

It is clear from the above arguments that these states define a consistent W-algebra that is a proper subalgebra of the tensor product of the two $N = 2$ algebras. It is thus natural to ask whether this property may be true for all generalised permutation branes. In order to understand how this generalisation could work it is first instructive to analyse the usual permutation branes.

3.6 The W-algebra of the usual permutation branes

If the levels of the two $N = 2$ algebras are equal, the usual permutation branes are characterised by the property that they preserve a large W-algebra $\mathcal{W}_{\rm per}$. This W-algebra is the subalgebra of the $(N = 2) \times (N = 2)$ algebra that consists of the states that are invariant under the permutation of the two $N = 2$ factors. Obviously, $\mathcal{W}_{\rm per}$ contains the diagonal $N = 2$ algebra as a subalgebra, and we therefore have

$$(N=2)_{\rm diag} \subset \mathcal{W}_{\rm per} \subset (N=2) \times (N=2)\ . \tag{3.39}$$

In order to understand the structure of $\mathcal{W}_{\rm per}$ let us decompose the tensor product of the two $N = 2$ vacuum representations with respect to the diagonal $N = 2$ algebra

$$\mathcal{H}^{(1)}_0 \otimes \mathcal{H}^{(2)}_0 = \mathcal{H}_{0,0} \oplus \mathcal{H}_{1,0} \oplus \mathcal{H}_{2,0} \oplus 2\mathcal{H}_{3,0} \oplus \mathcal{H}_{7/2,1} \oplus \mathcal{H}_{7/2,-1} \oplus 3\mathcal{H}_{4,0} \oplus \cdots\ . \tag{3.40}$$

Here $\mathcal{H}_{H,Q}$ denotes the irreducible representation of the diagonal $N = 2$ algebra with conformal weight H and charge Q, and we have assumed that k_1 and k_2 are generic. For example, the highest weight state of the diagonal $N = 2$ algebra of the representation $\mathcal{H}_{1,0}$ is

$$K^{(1)} \equiv |1,0\rangle = \Big(c_2 J^{(1)}_{-1} - c_1 J^{(2)}_{-1}\Big)\,\Omega\,, \tag{3.41}$$

while the highest weight vector of $\mathcal{H}_{2,0}$ is

$$\begin{aligned} K^{(2)} \equiv |2,0\rangle = &\frac{1}{3}\Big(c_2(1-c_2)\big(2L^{(1)}_{-2} - 3(J^{(1)}_{-1})^2\big) \\ &+ c_1(1-c_1)\big(2L^{(2)}_{-2} - 3(J^{(2)}_{-1})^2\big) \\ &- 6(1-c_1)(1-c_2)J^{(1)}_{-1}J^{(2)}_{-1}\Big)\Omega\,, \end{aligned} \tag{3.42}$$

and Ω denotes the vacuum vector. The W-algebra $\mathcal{W}_{\text{per}}$ now consists of those diagonal $N=2$ representations for which the highest weight state is symmetric under the exchange of the two $N=2$ algebras. Clearly $K^{(1)}$ is anti-symmetric (if $c_1=c_2$), while $K^{(2)}$ in (3.42) is symmetric (if $c_1=c_2$). Continuing in this fashion one finds that for generic $k_1=k_2$ the W-algebra $\mathcal{W}_{\text{per}}$ is

$$\mathcal{W}_{\text{per}} = \mathcal{H}_{0,0} \oplus \mathcal{H}_{2,0} \oplus \mathcal{H}_{3,0} \oplus \mathcal{H}_{7/2,1} \oplus \mathcal{H}_{7/2,-1} \oplus 2\mathcal{H}_{4,0} \oplus \cdots \ . \tag{3.43}$$

3.7 A W-algebra for the generalised permutation branes?

As is clear from the above, the decomposition (3.40) also holds if $k_1 \neq k_2$. Given that the matrix factorisation description of the generalised permutation branes is very similar to that of the usual permutation branes, one may suspect that also the generalised permutation branes preserve a large W-algebra $\mathcal{W}_{\text{gper}}$. This W-algebra must again contain the diagonal $N=2$ algebra, and we can therefore, as before, decompose it in terms of representations of the diagonal $N=2$ algebra. Since $\mathcal{W}_{\text{gper}}$ must be a subalgebra of the tensor product of the two $N=2$ algebras, only the representations in (3.40) may appear.

Unlike the situation where the levels are equal, we do not know *a priori* how to characterise $\mathcal{W}_{\text{gper}}$. However, we may analyse step by step whether a given $N=2$ representation may be part of $\mathcal{W}_{\text{gper}}$ or not. For example, it is easy to see that $\mathcal{H}_{1,0}$ cannot appear in $\mathcal{W}_{\text{gper}}$, since $K^{(1)}$, together with the diagonal $N=2$ algebra, generates the full tensor product algebra (upon taking (anti-)commutators).[4] On the other hand, one may at first think that $K^{(2)}$ could be part of $\mathcal{W}_{\text{gper}}$. However, as we shall now explain, this cannot be the case.

Using standard conformal field theory techniques we can express the modes of $K^{(2)}_n$ in terms of the two $N=2$ algebras

$$\begin{aligned} K^{(2)}_n = &\frac{1}{3}\Big(c_2(1-c_2)\big(2L^{(1)}_n - 3\sum_m :J^{(1)}_{m+n}J^{(1)}_{-m}:\big) \\ &\quad + c_1(1-c_1)\big(2L^{(2)}_n - 3\sum_m :J^{(2)}_{m+n}J^{(2)}_{-m}:\big) \\ &\quad - 6(1-c_1)(1-c_2)\sum_m :J^{(1)}_{m+n}J^{(2)}_{-m}: \Big) \ . \end{aligned} \tag{3.44}$$

If $K^{(2)}$ is part of $\mathcal{W}_{\text{gper}}$, then so is any state that can be obtained by the action of $K^{(2)}$ modes and diagonal $N=2$ modes from the vacuum. For example, we find that

$$G^{+}_{1/2}\, G^{-}_{1/2}\, K^{(2)}_0\, K^{(2)} = a\, J_{-1}\Omega + b\, K^{(1)} \,, \tag{3.45}$$

where

$$\begin{aligned} a &= \frac{8}{3(c_1+c_2)}(c_1+c_2-1)(c_1+c_2-2)(c_1-1)(c_2-1)c_1\, c_2 \,, \\ b &= -\frac{8}{c_1+c_2}(c_1-c_2)(c_1-1)(c_2-1)(c_1+c_2-1)^2 \ . \end{aligned} \tag{3.46}$$

[4]This is also the reason why the representation $\mathcal{H}_{1,0}$ cannot appear in the W-algebra $\mathcal{W}_{\text{per}}$, see (3.43) above.

Unless $b = 0$ it thus follows that (3.45) contains $K^{(1)}$, and hence $\mathcal{W}_{\rm gper}$ would in fact again be the full tensor product. On the other hand, it is clear that the generalised permutation branes cannot preserve the full tensor product symmetry. Thus we conclude that $K^{(2)}$ cannot be part of $\mathcal{W}_{\rm gper}$ unless $b = 0$

On the other hand, b only vanishes if either $c_1 = c_2$ ($k_1 = k_2$) or $c_i = 1$ ($k_i = 1$).[5] The former case is just the usual permutation case for which we have seen that $K^{(2)}$ is in fact part of $\mathcal{W}_{\rm per}$. On the other hand, if $k_i = 1$ the vector $K^{(2)}$ is simply a null-vector, and hence there is nothing to discuss. (Incidentally this is also what happens in the analysis of the $(k = 1) \times (k = 4)$ example.)

It is obviously conceivable that there exists an $N = 2$ primary field at higher conformal weight that can be consistently added to the diagonal $N = 2$ algebra to produce a proper subalgebra $\mathcal{W}_{\rm gper}$ of the tensor product of the two $N = 2$ algebras. However, given the above result for $K^{(2)}$ this seems unlikely to us. We therefore suspect that the generalised permutation branes are *not* characterised by the property that they preserve *any extended $N = 2$ symmetry.*

It would be very interesting to understand how the generalised permutation branes can be characterised in general. Given their simplicity from the matrix factorisation point of view they should be singled out by some special property, but as we have just seen, this does not seem to involve the symmetry they preserve. This suggests that there must be a different point of view from which this class of boundary states is preferred. A similar phenomenon also occurred for the usual permutation branes [10], where it was found that only a subset of the permutation factorisations corresponds in fact to permutation branes. (The other permutation factorisations thus correspond to branes that preserve less symmetry.) The different permutation factorisations (that have a uniform matrix factorisation description) therefore correspond to branes that cannot be uniformly characterised in terms of the symmetry they preserve.

4. The $\hat{P}$-branes

For the $k_1 = 1$, $k_2 = 4$ theory there is a second natural class of branes that define 'generalised permutation branes', but that do not correspond to the above matrix factorisations. These branes (which we shall denote as $\hat{P}$-branes in the following) have in fact a simple description in terms of the permutation orbifold of the $(k = 1)^3$ theory that is also equivalent to the $(k = 1) \times (k = 4)$ theory; for completeness we shall also briefly describe them from the various points of view.

4.1 The conformal field theory description

The $\hat{P}$-branes couple only to the sectors

$$(\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}) \,,\; (\tfrac{1}{2}\tfrac{1}{2}, \tfrac{5}{2}\tfrac{5}{2}) \,,\; (\tfrac{3}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{5}{2}) \,,\; (\tfrac{1}{2}\tfrac{3}{2}, \tfrac{5}{2}\tfrac{1}{2}) \,. \tag{4.1}$$

In the first two sectors there is a non-trivial overlap with the tensor product branes. The relevant combinations of Ishibashi states with respect to the diagonal $N = 2$ symmetry are

[5] Recall that $c = 1$ is the smallest possible value for the $N = 2$ minimal models.

then

$$
\begin{aligned}
|\tfrac{1}{2}\tfrac{1}{2},\tfrac{1}{2}\tfrac{1}{2}\rangle\!\rangle^{\hat{P}} &= |0,0\rangle\!\rangle + \sum_{m=1}^{\infty}\sum_{n=1}^{m}\cos\tfrac{2\pi(m+n)}{3}|m^2+n^2-mn,0\rangle\!\rangle \\
&\quad + \sum_{n=0}^{\infty}\big(|6n+\tfrac{11}{2},1\rangle\!\rangle + |6n+\tfrac{11}{2},-1\rangle\!\rangle\big) \qquad (4.2)\\
|\tfrac{1}{2}\tfrac{1}{2},\tfrac{5}{2}\tfrac{5}{2}\rangle\!\rangle^{\hat{P}} &= \sum_{m=1}^{\infty}\sum_{n=1}^{m}\cos\tfrac{2\pi(m+n)}{3}|m^2+n^2-mn,0\rangle\!\rangle \\
&\quad + \sum_{n=0}^{\infty}\big(|6n+\tfrac{5}{2},1\rangle\!\rangle + |6n+\tfrac{5}{2},-1\rangle\!\rangle\big)\ . \qquad (4.3)
\end{aligned}
$$

Their coefficients $B_{m,n}$ are thus of the form

$$
B^{\hat{P}}_{m,n} = \begin{cases} 1 & \text{if } m+n = 0 \bmod 3 \\ -\frac{1}{2} & \text{otherwise.} \end{cases} \qquad (4.4)
$$

These coefficients satisfy again the factorisation constraint (3.36). Since now $B_{m,n} = 1$ for $m+n = 0 \bmod 3$, these branes preserve a W-algebra that is different from the one for the P-branes.

In the other two sectors the branes overlap with the generalised permutation branes. The relevant combinations of Ishibashi states for the $\hat{P}$-branes are

$$
\begin{aligned}
|\tfrac{3}{2}\tfrac{1}{2},\tfrac{1}{2}\tfrac{5}{2}\rangle\!\rangle^{\hat{P}} = &\sum_{\substack{m=0\ n=0\\ m+1 \text{ and } n \text{ even}}}^{\infty\ \ \infty} |\tfrac{1}{3}+(m-n)^2+(m+1)n,0\rangle\!\rangle \\
&+ \sum_{\substack{m=0\ n=0\\ m+1 \text{ or } n \text{ odd}\\ m+n+\mathrm{Mod}(m,2)+\mathrm{Mod}(n,2)=0 \mod 3}}^{\infty\ \ \infty} |\tfrac{1}{3}+(m-n)^2+(m+1)n,0\rangle\!\rangle \\
&+ \sum_{\substack{m=0\ n=0\\ m+1 \text{ or } n \text{ odd}\\ m+n+\mathrm{Mod}(m,2)+\mathrm{Mod}(n,2)=1 \mod 3}}^{\infty\ \ \infty} \Big(-\frac{\sqrt{3}+1}{2}\Big)|\tfrac{1}{3}+(m-n)^2+(m+1)n,0\rangle\!\rangle \\
&+ \sum_{\substack{m=0\ n=0\\ m+1 \text{ or } n \text{ odd}\\ m+n+\mathrm{Mod}(m,2)+\mathrm{Mod}(n,2)=2 \mod 3}}^{\infty\ \ \infty} \Big(\frac{\sqrt{3}-1}{2}\Big)|\tfrac{1}{3}+(m-n)^2+(m+1)n,0\rangle\!\rangle\ , \qquad (4.5)
\end{aligned}
$$

and similarly for $|\tfrac{1}{2}\tfrac{3}{2},\tfrac{5}{2}\tfrac{1}{2}\rangle\!\rangle^{\hat{P}}$. Here, $\mathrm{Mod}(m,2)$ is 0 or 1 depending on whether m is even or odd, respectively. The corresponding boundary states read

$$
\|\hat{P}(j)\rangle\!\rangle \ = \ \sqrt{2}\cdot 3^{\frac{1}{4}}\Big(|\tfrac{1}{2}\tfrac{1}{2},\tfrac{1}{2}\tfrac{1}{2}\rangle\!\rangle^{\hat{P}} + |\tfrac{1}{2}\tfrac{1}{2},\tfrac{5}{2}\tfrac{5}{2}\rangle\!\rangle^{\hat{P}}\Big) + 3^{\frac{1}{4}}e^{2\pi ij/3}|\tfrac{3}{2}\tfrac{1}{2},\tfrac{1}{2}\tfrac{5}{2}\rangle\!\rangle^{\hat{P}} + 3^{\frac{1}{4}}e^{-2\pi ij/3}|\tfrac{1}{2}\tfrac{3}{2},\tfrac{5}{2}\tfrac{1}{2}\rangle\!\rangle^{\hat{P}}\ , \qquad (4.6)
$$

where $j = 0,1,2$.

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The self-overlap of these boundary states is given by

$$
\begin{aligned}
&\langle\!\langle \hat{P}(j) \| q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}} \| \hat{P}(j) \rangle\!\rangle \\
&\quad = \chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) \Big(\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q}) + 2\chi^{k=4}_{\frac{5}{2}\frac{1}{2}}(\tilde{q}) + 2\chi^{k=4}_{\frac{1}{2}\frac{5}{2}}(\tilde{q}) \Big) \\
&\qquad + \chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q}) \Big(\chi^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q}) + \chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q}) + 2\chi^{k=4}_{\frac{3}{2}\frac{3}{2}}(\tilde{q}) + 2\chi^{k=4}_{\frac{1}{2}\frac{5}{2}}(\tilde{q}) \Big) \\
&\qquad + \chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q}) \Big(\chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q}) + 2\chi^{k=4}_{\frac{3}{2}\frac{3}{2}}(\tilde{q}) + 2\chi^{k=4}_{\frac{5}{2}\frac{1}{2}}(\tilde{q}) \Big) \\
&\qquad + \sum_{\substack{m=1 \\ m+n \not\equiv 0 \bmod 3}}^{\infty} \sum_{n=1}^{m} \chi^{c=3,\mathrm{NS}}_{(\frac{1}{9}(m^2+n^2-mn),0)}(\tilde{q}) \\
&\quad = \big(\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) \chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q}) \chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q}) \chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q}) \big) \\
&\qquad \times \big(\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q}) \big) \\
&\qquad + \big(\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}^{\frac{1}{3}}) + \chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q}^{\frac{1}{3}}) + \chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q}^{\frac{1}{3}}) \big) \,. \qquad (4.7)
\end{aligned}
$$

In the last step we have used the identities of appendix B.3 to rewrite the spectrum in terms of $k = 1$-characters.

The $\hat{P}$-boundary states are in fact elementary although we are only considering the NSNS sector. This implies that they actually do not couple to *any* RR states. In particular, they therefore cannot carry any RR charges.

4.2 The matrix factorisation point of view

Since the $\hat{P}$-branes do not couple to the RR sector, the $\hat{P}$-branes coincide with their own anti-branes. In the matrix factorisation language this means that the corresponding factorisations must be equivalent to their reverse factorisations. We now propose that the relevant factorisations are

$$
E = \begin{pmatrix} (x-\xi y^2) & y^3 \\ -y^3 & \pi_\xi(x,y^2) \end{pmatrix}, \qquad J = \begin{pmatrix} \pi_\xi(x,y^2) & -y^3 \\ y^3 & (x-\xi y^2) \end{pmatrix}, \qquad (4.8)
$$

where we have used the notation

$$
\pi_\xi(v,w) = \prod_{\xi' \neq \xi} (v - \xi' w)\,, \qquad (4.9)
$$

and ξ is again a third root of -1 (see eq. (2.6)). It is very easy to see that this factorisation is indeed equivalent to its own reverse.

It follows from (4.7) that the self-spectrum of the $\hat{P}$-branes contains six bosonic (and six fermionic) states, whose $\mathrm{U}(1)$ charges are $0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 1$. This can now be compared with the self-spectrum of the corresponding matrix factorisation (4.8). Because the factorisation is equivalent to its own reverse, it is sufficient to consider only the bosons. The boson with $\mathrm{U}(1)$ charge 0 is simply the identity matrix, $\phi_0 = \phi_1 = \mathbf{1}$, and the boson with $\mathrm{U}(1)$ charge 1 can be represented by the off-diagonal matrix

$$
\phi_1 = \begin{pmatrix} 0 & (x-2y^2)y^2 \\ -3x-4y^2 & 0 \end{pmatrix}, \qquad (4.10)
$$

where, for simplicity, we have only given the result for $\xi = -1$. [The matrix ϕ_0 is then determined from ϕ_1 as $\phi_0 = \frac{1}{W} E \phi_1 J$.] For $\xi = -1$, the two bosons of charge $\frac{1}{3}$ can be described, up to exact solutions, by the matrix

$$\phi_1 = \begin{pmatrix} ay & b\,(-x+2y^2) \\ b & (a-3b)y \end{pmatrix} , \tag{4.11}$$

while the two bosons of charge $\frac{2}{3}$ can (again for $\xi = -1$) be described by the matrix

$$\phi_1 = \begin{pmatrix} ax & bxy \\ -by & ax + b(-2x+y^2) \end{pmatrix} . \tag{4.12}$$

The result for the other values of ξ is similar. This gives strong support to the claim that (4.8) is indeed the matrix factorisation of the $\hat{P}$-branes.

4.3 The permutation orbifold point of view

The $\hat{P}$-branes have in fact a simple description in terms of yet another realisation of the $A_1 \times A_4$ model, namely as a permutation orbifold of three minimal models at $k = 1$. The relevant triple product of $k = 1$ theories is the one that corresponds to the superpotential

$$W = W_{A_1 \times A_1 \times A_1'} = u^3 + v^3 + w^3 + z^2 . \tag{4.13}$$

Here A_1' denotes the A_1 minimal model with the opposite GSO projection [26] (see also [35]). Obviously, all the different branes we have discussed in this paper can also be described in terms of matrix factorisations of W in (4.13); this is discussed in appendix C.

To obtain the $A_1 \times A_4$ model from this theory, we have to perform a $\mathbb{Z}_2$-orbifold which acts as $v \leftrightarrow w$ together with $z \to -z$. Since the u^3 term is not involved, this just means that the $\mathbb{Z}_2$ orbifold of $A_1 \times A_1'$ is in fact equivalent to the A_4 model. This can, for example, be confirmed by checking that the relevant matrix factorisations are in one-to-one correspondence.

The conformal field theory corresponding to (4.13) is

$$\mathcal{H} = \bigoplus_{[l_i,m_i,s_i],s_i'} \bigotimes_{i=1}^{3} \mathcal{H}_{[l_i,m_i,s_i]} \otimes \mathcal{H}_{[l_i,m_i,s_i']} , \tag{4.14}$$

where $i = 1, 2, 3$ labels the sectors corresponding to the variables u, v, w. The sum over the s_i and s_i' is restricted such that they are either all odd or all even and by the requirement that

$$s_1 + s_2 + s_3 - s_1' - s_2' - s_3' = 0 \mod 4 . \tag{4.15}$$

We note in passing that this GSO projection is incompatible with the B-type gluing condition in the Ramond-Ramond sector. To see this we recall that the B-type condition requires the left- and right-moving $U(1)$-charge to be related as $Q_L = -Q_R$. Expressing the charges in terms of the coset labels (see appendix A), it follows that

$$\frac{2}{3}(m_1 + m_2 + m_3) - \frac{1}{2}(s_1 + s_2 + s_3 + s_1' + s_2' + s_3') = 0 \mod 2 . \tag{4.16}$$

Using the constraint (4.15) B-type gluing would require

$$\frac{2}{3}(m_1 + m_2 + m_3) = 1 \mod 2 \,. \tag{4.17}$$

For integer labels m_i this can never be achieved, and thus all B-type branes of this theory only couple to the NSNS sector.

The analogue of the above $\mathbb{Z}_2$-orbifold in the Landau–Ginzburg description is now the $\mathbb{Z}_2$-orbifold that acts as a transposition in the second and third factor. From the character decompositions in appendix B.3 this is fairly obvious. It is also not hard to see how the GSO-projections are related: the projection (4.15) allows e.g. the labels s_i to equal s_i'. The decomposition into $k = 4$ representations and restriction to the invariant part leaves us again with identical left- and right-moving labels s. On the $(k = 1) \times (k = 4)$ side we thus obtain the GSO condition

$$s_1 + S_2 - s_1' - S_2' = 0 \mod 4 \,, \tag{4.18}$$

where we denoted the coset label of the $(k = 4)$-part by (capital) S_2. This is the same GSO projection as in (2.2).

From the permutation orbifold point of view the $\hat{P}_j$-branes are simply the superposition of the two permutation boundary states $\|(12)_{\xi_j}\rangle\rangle$ and $\|(13)_{\xi_j}\rangle\rangle$. The self-spectrum of the $\hat{P}$-branes thus consists of the self-spectrum of a transposition brane,

$$\begin{aligned}\langle\langle(12)_{\xi_j}\|q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}}\|(12)_{\xi_j}\rangle\rangle = &\big(\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q})\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q}) \\ &+ \chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\big) \\ &\times \big(\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q}) + \chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\big)\end{aligned} \tag{4.19}$$

and the relative spectrum between two transposition branes,

$$\langle\langle(12)_{\xi_j}\|q^{\frac{1}{2}(L_0+\tilde{L}_0)-\frac{c}{24}}\|(13)_{\xi_j}\rangle\rangle = \chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q}^{\frac{1}{3}}) + \chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q}^{\frac{1}{3}}) + \chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q}^{\frac{1}{3}}) \,. \tag{4.20}$$

The $\tilde{q}^{\frac{1}{3}}$ powers appear since the relative permutation is of order three. This now agrees precisely with (4.7).

We also know that the $T(0,0)$ brane comes from the permutation brane $\|(23)_{\xi_0}\rangle\rangle$ in the $(k = 1)^3$ theory. It is invariant under the $\mathbb{Z}_2$ orbifold, and decomposes into two fractional branes corresponding to the $T(0,0)$ brane and its anti-brane. The spectrum between the $T(0,0)$ brane and a $\hat{P}$ brane is then just given by (4.20). Finally, it is also not difficult to construct these branes on the torus — they correspond to superpositions of D1-branes at angles ϕ_n. In contradistinction to the P-branes, these D1-branes now do not go through any fixed points and therefore need not be resolved.

5. Conclusions

In this paper we have constructed the generalised $N = 2$ permutation branes for the simple example of the product theory with levels $k = 1$ and $k = 4$. This was done by analysing the product theory with respect to the diagonal $N = 2$ algebra. In particular, we classified all the B-type Ishibashi states with respect to the diagonal $N = 2$ algebra, and then constructed the generalised permutation boundary states by taking suitable combinations of these Ishibashi states (see (2.34)). We then checked that these P-brane boundary states reproduce the correct (topological) open string spectrum. We also showed that they correspond to certain standard D-branes in the $\mathbb{Z}_6$-orbifold of the $su(3)$ torus, thus giving strong support for their consistency. Finally, we also analysed a second class of $\hat{P}$-brane boundary states, and explained how they can be naturally described from the point of view of the permutation orbifold of the product theory of three $k = 1$ models.

As we mentioned in the introduction, these D-branes play an important role in accounting for the RR charges of Gepner models. Indeed, the above constructions appear for 3 of the 31 A-type Gepner models for which generalised permutation branes are required: these are the models $\mathbb{P}_{(3,3,4,6,8)}[24]$, $\mathbb{P}_{(3,5,6,6,10)}[30]$ and $\mathbb{P}_{(3,10,12,15,20)}[60]$ (see table 2 of [11]). For these Gepner models our construction therefore gives a conformal field theory description of the missing D-branes.

It would obviously be very interesting to generalise the construction to the other cases of interest. Clearly, the case of $c = 3$ is special since we have a free field realisation (as well as the relation to the triple product theory), but the first approach (section 2) should also be possible in the general case. In particular, the spectral flow symmetry should still be useful in constraining the coefficients in front of the different Ishibashi states. On the other hand, as we have argued in section 3.7, it seems unlikely that the generalised permutation branes preserve any extended $N = 2$ symmetry, and it may therefore be difficult to guess the correct ansatz.

It is clear that the generalised permutation branes are very special from the point of view of the matrix factorisation analysis. It would therefore be very interesting to understand how to characterise these branes intrinsically in conformal field theory. This will probably require a new way of looking at D-branes. One natural guess would be that these branes are singled out by special properties of their open string spectra. As we have seen in the $(k = 1) \times (k = 4)$ case, the open string spectra look often much simpler than the formulae for the boundary states. For example, the spectrum of a single P-brane can still be written in terms of characters of the full product algebra, although the boundary states of the P-branes explicitly break the full $(N = 2) \times (N = 2)$ symmetry. It would be very interesting to understand which properties of the open string spectra will generalise.

The $N = 2$ minimal models can be described as cosets of SU(2) theories, and the generalised permutation branes of the $N = 2$ theories should therefore be closely related to the generalised permutation branes for products of SU(2) group manifolds [36]. In the group case their existence was argued for on the level of the effective world-volume theory.[6] Given the close relation between the $N = 2$ algebra and SU(2) one might then expect that one can construct the generalised permutation branes for the product of two SU(2) WZW models at $k = 1$ and $k = 4$ by combining our $N = 2$ construction with suitable boundary conditions for the U(1) factors. The resulting branes on the two SU(2)s factorise into the

[6]A first step to perform a similar analysis for coset theories was also recently taken in [37].

coset and a U(1) part, and one can show that they do not preserve the diagonal SU(2) symmetry. This symmetry, however, is expected from the analysis of [36], and it thus seems that we do not obtain the generalised permutation branes of [36] in this manner. Nevertheless, we expect that the two constructions are in fact closely related; it would be very interesting to understand this better.

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A. Conventions

A $N=2$ minimal model of level k has central charge

$$c = \frac{3k}{k+2} \; .$$

These models can be described by a coset construction. The sectors of the coset theory are labelled by three integers (l,m,s) where $l = 0,\ldots,k$ and m and s are defined modulo $2(k+2)$ and 4, respectively. Here $l+m+s$ is even, and we have the field identifications $(l,m,s) \sim (k-l, m+k+2, s+2)$. These sectors are representations of the bosonic subalgebra of the $N=2$ superconformal algebra.

In the following we will often use a different notation to label the full representations of the $N=2$ algebra. The Neveu-Schwarz sectors are labelled by two half-integers u,v, where $0 \leq u,v,u+v \leq k+1$. Conformal weight h and $U(1)$ charge q are given by

$$h^{\mathrm{NS}}_{(u,v)} = \frac{uv-1/4}{k+2} \qquad q^{\mathrm{NS}}_{(u,v)} = \frac{u-v}{k+2} \; . \tag{A.1}$$

The relation to the coset sectors is

$$(u,v) \equiv (l,m,0) \oplus (l,m,2) \, , \tag{A.2}$$

where

$$l = u+v-1 \qquad m = v-u \; . \tag{A.3}$$

The Ramond sectors are labelled by two integers u,v with the range $0 \leq v \leq k$ and $1 \leq u \leq k+1-v$. The conformal weight and charge is

$$h^{\mathrm{R}}_{(u,v)} = \frac{uv}{k+2} + \frac{c}{24} \qquad q^{\mathrm{R}}_{(u,v)} = \frac{u-v}{k+2} - \frac{1}{2} \; . \tag{A.4}$$

The relation to the coset sectors is

$$(u,v) \;\equiv\; (l,m,-1) \oplus (l,m,1) \, , \tag{A.5}$$

where again

$$l = u + v - 1 \qquad\qquad m = v - u \ . \tag{A.6}$$

The representations with $v = 0$ have Ramond ground states of charge $q^{\rm R}_{(u,v)}$. For the other representations, the charges of the lowest lying states are $q^{\rm R}_{(u,v)}$ and $q^{\rm R}_{(u,v)} + 1$.

The $N = 2$ algebra possesses a family of outer automorphisms, usually referred to as spectral flow, that are parametrised by t. They act on the generators of the algebra as

$$\begin{aligned} \alpha_t(L_n) &= L_n - t\, J_n + \tfrac{t^2 c}{6}\, \delta_{n,0} \\ \alpha_t(J_n) &= J_n - \tfrac{tc}{3}\, \delta_{n,0} \\ \alpha_t(G^{\pm}_r) &= G^{\pm}_{r \mp t} \ . \end{aligned} \tag{A.7}$$

If t is half-integer, the spectral flow connects Neveu-Schwarz- and Ramond-representations, whereas integer t maps Neveu-Schwarz (Ramond) representations to themselves. It follows immediately from the above that the conformal weight and charge of any state transforms as

$$(h, q) \rightarrow (h - t\, q + \tfrac{t^2 c}{6}, q - \tfrac{tc}{3}) \ . \tag{A.8}$$

In general, a highest weight state of the original $N = 2$ algebra is however not highest weight with respect to the spectrally flowed algebra; thus one cannot directly read off from (A.8) the weight and charge of the highest weight state of the new (spectrally flowed) representation. In the generic case (when there is no 'accidental null vector'), the representation (h_t, q_t) that is reached by spectral flow is

$$(h_t, q_t) \ = \ (h + t^2(\tfrac{c}{6} - \tfrac{1}{2}) - t\, q, q - t(\tfrac{c}{3} - 1)) \ . \tag{A.9}$$

For $c = 3$ this simplifies to the formula (valid in the generic case)

$$(h_t, q_t) \ = \ (h - t\, q, q) \ . \tag{A.10}$$

For example, for $t = -3$, this then maps the representation $(6n + \frac{5}{2}, 1) \mapsto (6n + \frac{11}{2}, 1)$. On the other hand, (A.10) does not always hold; for example, for the case of the vacuum representation, there is an 'accidental' null vector so that the original ground state is also highest weight with respect to the $t = -1$ algebra. Thus under $t = -1$, we have $(0, 0) \mapsto (\frac{1}{2}, 1)$, leading to the flow $(0, 0) \mapsto (\frac{5}{2}, 1)$ for $t = -3$.

For the $k^{\rm th}$ minimal model, it is easy to see that spectral flow by $t = \pm(k + 2)$ acts trivially on all representations. Thus we may restrict to $0 \leq t < k + 2$, for which we find

$$(u, v) \rightarrow \begin{cases} (u + t, v - t) & \text{for } v \geq t \\ (t - v, k + 2 - u - t) & \text{for } v < t \ . \end{cases} \tag{A.11}$$

B. Decomposition of characters

B.1 $k_1 = 1$, $k_2 = 4$, Neveu-Schwarz sector

In this appendix we list the decomposition of characters of the tensor product of two minimal models with level $k_1 = 1$ (central charge $c_1 = 1$) and with level $k_2 = 4$ ($c_2 = 2$)

into characters of the diagonal $N = 2$ algebra. The labelling of the sectors is done by half-integers u_1, v_1, u_2, v_2; the conformal weights and charges are determined as in (A.1). The decomposition of the characters is given by

$$\begin{aligned}
\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(z,q) &= \chi^{c=3,\mathrm{NS}}_{(0,0)}(z,q) + \sum_{m=1}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{((m-n)^2+mn,0)}(z,q) \\
&\quad + \sum_{n=0}^{\infty}\left(\chi^{c=3,\mathrm{NS}}_{(6n+\frac{11}{2},1)}(z,q) + \chi^{c=3,\mathrm{NS}}_{(6n+\frac{11}{2},-1)}(z,q)\right) \qquad \text{(B.1)}\\
\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{3}{2}\frac{3}{2}}(z,q) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\chi^{c=3,\mathrm{NS}}_{(\frac{1}{3}+(m-n)^2+(m+1)n,0)}(z,q) \qquad \text{(B.2)}\\
\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(z,q) &= \sum_{m=1}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{((m-n)^2+mn,0)}(z,q) \\
&\quad + \sum_{n=0}^{\infty}\left(\chi^{c=3,\mathrm{NS}}_{(\frac{5}{2}+6n,1)}(z,q) + \chi^{c=3,\mathrm{NS}}_{(\frac{5}{2}+6n,-1)}(z,q)\right) \qquad \text{(B.3)}\\
\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{1}{2}\frac{5}{2}}(z,q) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\chi^{c=3,\mathrm{NS}}_{(\frac{1}{3}+(m-n)^2+(m+1)n,0)}(z,q) \qquad \text{(B.4)}\\
\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(z,q) &= \sum_{m=1}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{((m-n)^2+mn,0)}(z,q) \\
&\quad + \sum_{n=0}^{\infty}\left(\chi^{c=3,\mathrm{NS}}_{(\frac{3}{2}+6n,-1)}(z,q) + \chi^{c=3,\mathrm{NS}}_{(\frac{7}{2}+6n,1)}(z,q)\right) \qquad \text{(B.5)}\\
\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(z,q) &= \sum_{n=0}^{\infty}\left(\chi^{c=3,\mathrm{NS}}_{(\frac{1}{2}+6n,1)}(z,q) + \chi^{c=3,\mathrm{NS}}_{(\frac{9}{2}+6n,-1)}(z,q)\right) \\
&\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{((m-n)^2+mn,0)}(z,q)\ . \qquad \text{(B.6)}
\end{aligned}$$

These are the ones that are relevant for B-type boundary conditions. The others are

$$\begin{aligned}
\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(z,q) &= \sum_{n=0}^{\infty}\left(\chi^{c=3,\mathrm{NS}}_{(\frac{1}{6}+2n,\frac{1}{3})}(z,q) + \chi^{c=3,\mathrm{NS}}_{(\frac{1}{6}+\frac{7}{2}+4n,-\frac{2}{3})}(z,q)\right) \qquad \text{(B.7)}\\
\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{3}{2}\frac{1}{2}}(z,q) &= \sum_{n=0}^{\infty}\left(\chi^{c=3,\mathrm{NS}}_{(\frac{1}{12}+n,\frac{1}{6})}(z,q) + \chi^{c=3,\mathrm{NS}}_{(\frac{1}{12}+\frac{9}{2}+5n,-\frac{5}{6})}(z,q)\right) \qquad \text{(B.8)}\\
\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{5}{2}\frac{1}{2}}(z,q) &= \sum_{n=0}^{\infty}\left(\chi^{c=3,\mathrm{NS}}_{(\frac{1}{6}+n,\frac{1}{3})}(z,q) + \chi^{c=3,\mathrm{NS}}_{(\frac{1}{6}+\frac{3}{2}+2n,-\frac{2}{3})}(z,q)\right) \qquad \text{(B.9)}\\
\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{7}{2}\frac{1}{2}}(z,q) &= \sum_{n=0}^{\infty}\left(\chi^{c=3,\mathrm{NS}}_{(\frac{1}{4}+3n,\frac{1}{2})}(z,q) + \chi^{c=3,\mathrm{NS}}_{(\frac{1}{4}+1+3n,\frac{1}{2})}(z,q)\right. \\
&\quad \left. + \chi^{c=3,\mathrm{NS}}_{(\frac{1}{4}+\frac{3}{2}+3n,-\frac{1}{2})}(z,q) + \chi^{c=3,\mathrm{NS}}_{(\frac{1}{4}+\frac{5}{2}+3n,-\frac{1}{2})}(z,q)\right) \qquad \text{(B.10)}\\
\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(z,q) &= \sum_{n=0}^{\infty}\left(\chi^{c=3,\mathrm{NS}}_{(\frac{1}{3}+4n,\frac{2}{3})}(z,q) + \chi^{c=3,\mathrm{NS}}_{(\frac{1}{3}+\frac{3}{2}+2n,-\frac{1}{3})}(z,q)\right) \qquad \text{(B.11)}
\end{aligned}$$

$$\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{5}{2}\frac{3}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{7}{12}+n,\frac{1}{6})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{7}{12}+\frac{3}{2}+5n,-\frac{5}{6})}(z,q)\Big) \tag{B.12}$$

$$\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{5}{6}+2n,\frac{1}{3})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{5}{6}+\frac{3}{2}+4n,-\frac{2}{3})}(z,q)\Big) \tag{B.13}$$

$$\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{3}{2}\frac{3}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{1}{2}+n,\frac{1}{3})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{1}{2}+\frac{1}{2}+2n,-\frac{2}{3})}(z,q)\Big) \tag{B.14}$$

$$\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{7}{6}+2n,\frac{1}{3})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{7}{6}+\frac{1}{2}+4n,-\frac{2}{3})}(z,q)\Big) \tag{B.15}$$

$$\begin{aligned}\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{3}{2}\frac{1}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(&\chi^{c=3,\mathrm{NS}}_{(\frac{1}{4}+3n,\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{1}{4}+2+3n,\frac{1}{2})}(z,q)\\ &+\chi^{c=3,\mathrm{NS}}_{(\frac{1}{4}+\frac{1}{2}+3n,-\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{1}{4}+\frac{5}{2}+3n,-\frac{1}{2})}(z,q)\Big)\end{aligned} \tag{B.16}$$

$$\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{1}{2}\frac{3}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{1}{4}+n,\frac{1}{6})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{1}{4}+\frac{7}{2}+5n,-\frac{5}{6})}(z,q)\Big) \tag{B.17}$$

$$\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{5}{2}\frac{1}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{1}{3}+2n,\frac{2}{3})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{1}{3}+\frac{1}{2}+n,-\frac{1}{3})}(z,q)\Big) \tag{B.18}$$

$$\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{7}{2}\frac{1}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{5}{12}+5n,\frac{5}{6})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{5}{12}+\frac{1}{2}+n,-\frac{1}{6})}(z,q)\Big) \tag{B.19}$$

$$\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{1}{2}\frac{7}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{5}{12}+n,-\frac{1}{6})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{5}{12}+\frac{5}{2}+5n,\frac{5}{6})}(z,q)\Big) \tag{B.20}$$

$$\begin{aligned}\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{5}{2}\frac{3}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(&\chi^{c=3,\mathrm{NS}}_{(\frac{3}{4}+3n,\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{3}{4}+1+3n,\frac{1}{2})}(z,q)\\ &+\chi^{c=3,\mathrm{NS}}_{(\frac{3}{4}+\frac{1}{2}+3n,-\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{3}{4}+\frac{3}{2}+3n,-\frac{1}{2})}(z,q)\Big)\end{aligned} \tag{B.21}$$

$$\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{3}{2}\frac{5}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{3}{4}+n,\frac{1}{6})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{3}{4}+\frac{1}{2}+5n,-\frac{5}{6})}(z,q)\Big) \tag{B.22}$$

$$\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(1+4n,\frac{2}{3})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(1+\frac{1}{2}+2n,-\frac{1}{3})}(z,q)\Big) \tag{B.23}$$

$$\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{1}{2}\frac{9}{2}}(z,q) = \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{1}{2}+2n,-\frac{1}{3})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{1}{2}+\frac{5}{2}+4n,\frac{2}{3})}(z,q)\Big)\ . \tag{B.24}$$

There are some identities between these products of characters which will be useful later, namely

$$\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\Big(\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(z,q)+\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(z,q)\Big) = \chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{5}{2}\frac{1}{2}}(z,q) \tag{B.25}$$

$$\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\Big(\chi^{k=4}_{\frac{1}{2}\frac{9}{2}}(z,q)+\chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(z,q)\Big) = \chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(z,q)\chi^{k=4}_{\frac{3}{2}\frac{3}{2}}(z,q) \tag{B.26}$$

$$\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\Big(\chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(z,q)+\chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(z,q)\Big) = \chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{5}{2}\frac{1}{2}}(z,q) \tag{B.27}$$

$$\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{3}{2}\frac{3}{2}}(z,q)=\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{1}{2}\frac{5}{2}}(z,q) = \chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(z,q)\chi^{k=4}_{\frac{5}{2}\frac{1}{2}}(z,q). \quad \text{(B.28)}$$

Furthermore we have

$$\begin{aligned} 6\sum_{n\geq 1}\sum_{m\geq n}\chi^{c=3,\mathrm{NS}}_{(m^2+n^2-mn,0)}(z,q)+\frac{\vartheta_3(z,q)}{\eta^3(q)} =& \chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\Big(\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(z,q)+\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(z,q)\Big) \\ &+\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\Big(\chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(z,q)+\chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(z,q)\Big) \\ &+\chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(z,q)\Big(\chi^{k=4}_{\frac{1}{2}\frac{9}{2}}(z,q)+\chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(z,q)\Big)\ . \end{aligned} \quad \text{(B.29)}$$

B.2 $k_1 = 1, k_2 = 4$, Ramond sector

The decompositions of products of NS characters in the preceding section can (by spectral flow) be used to write down the decomposition of products of R characters. We will distinguish $c=3$-representations corresponding to Ramond ground states by ${}^{\mathrm{g}}\chi^{c=3,\mathrm{R}}_{(\frac{1}{8},q)}(z,q)$, for the others we will use the notation $\chi^{c=3,\mathrm{R}}_{(h,q)}(z,q)$ where the charges of the lowest lying states are $q\pm\frac{1}{2}$.
The decompositions which are relevant for B-type boundary conditions are

$$\begin{aligned} \chi^{k=1}_{1\,0}(z,q)\chi^{k=4}_{4\,0}(z,q) \;=&\; {}^{\mathrm{g}}\chi^{c=3,\mathrm{R}}_{(\frac{1}{8},0)}(z,q)+\sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+3+3n,\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+1+3n,\frac{1}{2})}(z,q) \\ &+\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+2+3n,-\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+3+3n,-\frac{1}{2})}(z,q)\Big) \end{aligned} \quad \text{(B.30)}$$

$$\begin{aligned} \chi^{k=1}_{2\,0}(z,q)\chi^{k=4}_{2\,0}(z,q) \;=&\; {}^{\mathrm{g}}\chi^{c=3,\mathrm{R}}_{(\frac{1}{8},0)}(z,q)+\sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+3+3n,\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+2+3n,\frac{1}{2})}(z,q) \\ &+\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+1+3n,-\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+3+3n,-\frac{1}{2})}(z,q)\Big) \end{aligned} \quad \text{(B.31)}$$

$$\begin{aligned} \chi^{k=1}_{2\,0}(z,q)\chi^{k=4}_{3\,1}(z,q) \;=&\; \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+3n,\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+1+3n,\frac{1}{2})}(z,q) \\ &+\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+1+3n,-\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+2+3n,-\frac{1}{2})}(z,q)\Big) \end{aligned} \quad \text{(B.32)}$$

$$\begin{aligned} \chi^{k=1}_{1\,1}(z,q)\chi^{k=4}_{2\,2}(z,q) \;=&\; \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+1+3n,-\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+2+3n,-\frac{1}{2})}(z,q) \\ &+\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+1+3n,\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{1}{8}+2+3n,\frac{1}{2})}(z,q)\Big) \end{aligned} \quad \text{(B.33)}$$

$$\begin{aligned} \chi^{k=1}_{1\,1}(z,q)\chi^{k=4}_{1\,1}(z,q) \;=&\; \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+3n,-\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+2+3n,-\frac{1}{2})}(z,q) \\ &+\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+3n,\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+2+3n,\frac{1}{2})}(z,q)\Big) \end{aligned} \quad \text{(B.34)}$$

$$\begin{aligned} \chi^{k=1}_{1\,0}(z,q)\chi^{k=4}_{1\,3}(z,q) \;=&\; \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+3n,-\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+1+3n,-\frac{1}{2})}(z,q) \\ &+\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+1+3n,\frac{1}{2})}(z,q)+\chi^{c=3,\mathrm{R}}_{(\frac{5}{8}+2+3n,\frac{1}{2})}(z,q)\Big)\ . \end{aligned} \quad \text{(B.35)}$$

B.3 Relations between minimal models with level 1 and 4

The $N=2$ minimal model with central charge $c=1$ ($k=1$) has 3 sectors:

$$(\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{3}{2}),(\tfrac{3}{2},\tfrac{1}{2})\ . \tag{B.36}$$

Products of two of these characters can be decomposed in terms of the diagonal $N=2$ algebra with central charge $c=2$:

$$\begin{aligned}
\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q) &= \chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(z,q)+\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(z,q) && \text{(B.37)}\\
\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q) &= \chi^{k=4}_{\frac{5}{2}\frac{1}{2}}(z,q) && \text{(B.38)}\\
\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q) &= \chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(z,q)+\chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(z,q) && \text{(B.39)}\\
\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(z,q) &= \chi^{k=4}_{\frac{3}{2}\frac{3}{2}}(z,q)\ . && \text{(B.40)}
\end{aligned}$$

From this it is easy to see that the D-type minimal model at level 4 is equivalent to the product of two (A-type) minimal models at level 1.

Let us note further relations between characters:

$$\begin{aligned}
\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z^3,q^3) &= \chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\big(\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(z,q)+\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(z,q)\big)\\
&\qquad -3\sum_{\substack{m=1\\ m+n\not\equiv 0 \bmod 3}}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{(m^2+n^2-mn,0)}(z,q)
\end{aligned}\tag{B.41}$$

$$\begin{aligned}
\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q^{\frac{1}{3}}) &= \sum_{n=0}^{\infty}\Big(\chi^{c=3,\mathrm{NS}}_{(\frac{1+2n}{6},\frac{1}{3})}(z,q)+\chi^{c=3,\mathrm{NS}}_{(\frac{1+2n}{3},-\frac{2}{3})}(z,q)\Big)\\
&= \chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{5}{2}\frac{1}{2}}(z,q)+\chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(z,q)\chi^{k=4}_{\frac{1}{2}\frac{5}{2}}(z,q)\\
&\qquad +\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(z,q)\chi^{k=4}_{\frac{3}{2}\frac{3}{2}}(z,q)
\end{aligned}\tag{B.42}$$

$$\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(z,q^{\frac{1}{3}}) = \sum_{\substack{m=1\\ m+n\not\equiv 0 \bmod 3}}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{(\frac{m^2+n^2-mn}{9},0)}(z,q)\ . \tag{B.43}$$

B.4 Some character identities

We use the usual conventions for the theta functions:

$$\begin{aligned}
\vartheta_1(z,q) &= -i\,q^{1/12}\,\eta(q)\,(z^{1/2}-z^{-1/2})\prod_{n=1}^{\infty}(1-z\,q^n)(1-z^{-1}\,q^n)\\
\vartheta_2(z,q) &= q^{1/12}\,\eta(q)\,(z^{1/2}+z^{-1/2})\prod_{n=1}^{\infty}(1+z\,q^n)(1+z^{-1}\,q^n)\\
\vartheta_3(z,q) &= q^{-1/24}\,\eta(q)\prod_{n=1}^{\infty}(1+z\,q^{n-1/2})(1+z^{-1}\,q^{n-1/2})\\
\vartheta_4(z,q) &= q^{-1/24}\,\eta(q)\prod_{n=1}^{\infty}(1-z\,q^{n-1/2})(1-z^{-1}\,q^{n-1/2})\\
\eta(q) &= q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\ .
\end{aligned}$$

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Finally, we collect some identities relating theta functions to characters of the $k=1$ and $k=4$ $N=2$ minimal models:

$$\begin{aligned}\frac{1}{2}\sum_{p\in\Lambda_R}\tilde{q}^{\frac{1}{2}p^2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})} =& \frac{1}{2}\Big(\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})\big(\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})+\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q})\big)\\ &+\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q})\big(\chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})+\chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q})\big)\\ &+\chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\big(\chi^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q})+\chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q})\big)\Big)\end{aligned} \tag{B.44}$$

$$\frac{1}{2}\sum_{p\in\Lambda_R}\tilde{q}^{\frac{1}{2}(p+u_1)^2}\frac{\vartheta_3(\tilde{q})}{\eta^3(\tilde{q})} = \sum_{\substack{m=1\\ m\text{ or }n\text{ odd}}}^{\infty}\sum_{n=1}^{m}\chi^{c=3,\mathrm{NS}}_{(\frac{m^2+n^2-mn}{4},0)}(\tilde{q}) \tag{B.45}$$

$$\begin{aligned}\frac{\vartheta_3(\tilde{q})}{\vartheta_2(\tilde{q})} =& \frac{1}{2}\Big(\tilde{\chi}^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})\big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})-\tilde{\chi}^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q})\big)\\ &+\tilde{\chi}^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q})\big(\tilde{\chi}^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})+\tilde{\chi}^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q})\big)\\ &+\tilde{\chi}^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q})+\tilde{\chi}^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q})\big)\Big)\end{aligned} \tag{B.46}$$

$$\begin{aligned}-i\frac{\vartheta_3(\tilde{q}^{-\frac{1}{3}},\tilde{q})}{\vartheta_1(\tilde{q}^{-\frac{1}{3}},\tilde{q})} =& \frac{1}{2}\Big(\chi^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})\big(\chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})+\chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q})+\chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q})+\chi^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q})\big)\\ &+\chi^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q})\big(\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})+\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q})+\chi^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q})+\chi^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q})\big)\\ &+\chi^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\big(\chi^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})+\chi^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q})+\chi^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q})+\chi^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})\big)\Big)\end{aligned} \tag{B.47}$$

$$\begin{aligned}\frac{\vartheta_3(\tilde{q}^{-\frac{1}{3}},\tilde{q})}{\vartheta_2(\tilde{q}^{-\frac{1}{3}},\tilde{q})} =& \frac{1}{2}\Big(\tilde{\chi}^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})\big(\tilde{\chi}^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})+\tilde{\chi}^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q})+\tilde{\chi}^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q})+\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q})\big)\\ &+\tilde{\chi}^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{q})\big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})-\tilde{\chi}^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q})-\tilde{\chi}^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{q})-\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{q})\big)\\ &+\tilde{\chi}^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{q})\big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{q})-\tilde{\chi}^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{q})-\tilde{\chi}^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{q})-\tilde{\chi}^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{q})\big)\Big)\end{aligned} \tag{B.48}$$

$$\begin{aligned}&\tilde{z}^{-\frac{1}{3}}\frac{\vartheta_3(\tilde{z}\tilde{q}^{-1/3},\tilde{q})}{\vartheta_2(\tilde{q}^{-1/3},\tilde{q})}-\tilde{z}^{\frac{1}{3}}\frac{\vartheta_3(\tilde{z}^{-1}\tilde{q}^{-1/3},\tilde{q})}{\vartheta_2(\tilde{q}^{-1/3},\tilde{q})}\\ &=\tilde{\chi}^{k=1}_{\frac{1}{2}\frac{3}{2}}(\tilde{z},\tilde{q})\Big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{z},\tilde{q})+\tilde{\chi}^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{z},\tilde{q})-\tilde{\chi}^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{z},\tilde{q})+\tilde{\chi}^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{z},\tilde{q})\Big)\\ &\quad-\tilde{\chi}^{k=1}_{\frac{3}{2}\frac{1}{2}}(\tilde{z},\tilde{q})\Big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{1}{2}}(\tilde{z},\tilde{q})+\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{z},\tilde{q})-\tilde{\chi}^{k=4}_{\frac{5}{2}\frac{5}{2}}(\tilde{z},\tilde{q})+\tilde{\chi}^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{z},\tilde{q})\Big)\\ &\quad-\tilde{\chi}^{k=1}_{\frac{1}{2}\frac{1}{2}}(\tilde{z},\tilde{q})\Big(\tilde{\chi}^{k=4}_{\frac{1}{2}\frac{9}{2}}(\tilde{z},\tilde{q})+\tilde{\chi}^{k=4}_{\frac{7}{2}\frac{3}{2}}(\tilde{z},\tilde{q})-\tilde{\chi}^{k=4}_{\frac{3}{2}\frac{7}{2}}(\tilde{z},\tilde{q})-\tilde{\chi}^{k=4}_{\frac{9}{2}\frac{1}{2}}(\tilde{z},\tilde{q})\Big)\,.\end{aligned} \tag{B.49}$$

C. Matrix factorisations of the permutation orbifold

In order to understand the relation between the D-branes of the $A_1\times A_4$ model and the orbifold of the $A_1\times A_1\times A_1'$ theory, we now study some simple matrix factorisations of the latter theory. The simplest factorisation is of permutation type in the factors v, w and tensor in the rest,

$$Q_{u(vw)z;\xi} = \begin{pmatrix} 0 & u \\ u^2 & 0 \end{pmatrix} \hat{\otimes}\, Q_{(vw);\xi} \,\hat{\otimes}\, Q_z \ . \tag{C.1}$$

Here, Q_z is the simplest factorisation of the z-part, while $Q_{(vw);\xi}$ is the permutation factorisation,

$$Q_z = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix} \ , \qquad Q_{(vw);\xi} = \begin{pmatrix} 0 & (v-\xi w) \\ \pi_\xi(v,w) & 0 \end{pmatrix} \ , \tag{C.2}$$

where π_ξ is defined as in (4.9). The symbol $\hat{\otimes}$ denotes the tensoring of matrix factorisations [17] that was explained in some detail in [19].

The behaviour of this factorisation under the $\mathbb{Z}_2$-orbifold action depends on the value of ξ. If $\xi = -1$, then Q transforms under the $\mathbb{Z}_2$-action (that exchanges v and w and maps $z \mapsto -z$) to an equivalent factorisation Q',

$$Q_{u(vw)z;\xi=-1} = \omega \ Q'_{u(vw)z;\xi=-1} \ \omega^{-1} \ . \tag{C.3}$$

There are two choices for ω which we denote by $\omega_\pm$,

$$\omega_\pm = \pm \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{C.4}$$

In order to determine the self-spectrum of this factorisation we note that before taking the orbifold, we have 4 bosons and 4 fermions. In the orbifold theory we are only allowed to keep the equivariant morphisms, so we have to determine how the morphisms coming from the different factors behave under the $\mathbb{Z}_2$-action. From the u-part there is one boson and one fermion, both of which are even under the $\mathbb{Z}_2$-action. The (vw)-part has 2 bosons of which one is even and the other is odd; it does not have any fermions. The z-part has one even boson and one odd fermion. Tensoring those together and restricting to the overall even morphisms we find in total 2 bosons and 2 fermions. This is the same spectrum that we have found for the $T(0,0)$ tensor brane in the $A_1 \times A_4$-model. The two choices $\omega_\pm$ correspond to brane and anti-brane.

For $\xi = e^{\pm\frac{i\pi}{6}}$, the image under the orbifold action $Q'_{u(vw)z;\xi}$ is equivalent to $Q_{u(vw)z;\bar{\xi}}$; in order to obtain an equivariant factorisation we therefore have to take the superposition of the two factorisations. This leads then to the $T(0,2)$ tensor brane in the $A_1 \times A_4$-model.

Finally, in order to obtain the $T(0,1)$-brane we consider the tensor product factorisation in the $A_1 \times A_1 \times A_1'$ model

$$Q_{\mathrm{t}} = \begin{pmatrix} 0 & u \\ u^2 & 0 \end{pmatrix} \hat{\otimes} \begin{pmatrix} 0 & 0 & v & w \\ 0 & 0 & -w^2 & v^2 \\ v^2 & -w & 0 & 0 \\ w^2 & v & 0 & 0 \end{pmatrix} \hat{\otimes}\, Q_z \quad \text{with} \quad \omega_\pm = \pm \mathbf{1}_2 \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{C.5}$$

From the v,w-part we get two even bosons and two odd fermions. Tensoring this with the other morphisms we find in total four bosons and four fermions. This is indeed what we expect for the $T(0,1)$ tensor brane in the $A_1 \times A_4$-model (see e.g. [10]). The two choices of ω correspond to brane and anti-brane.

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C.1 The matrix factorisations for $\hat{P}$ and P

This accounts for the tensor product branes of the $A_1 \times A_4$-model. The $\hat{P}$-branes are now related to factorisations of the $A_1 \times A_1 \times A_1'$ theory that involve a permutation of the u-factor. The simplest such factorisation that is equivariant with respect to the $\mathbb{Z}_2$-action involves a superposition of a (uv)-permutation and a (uw)-permutation with the same choice for ξ,

$$Q_{\hat{P}} = \begin{pmatrix} 0 & 0 & u-\xi v & w \\ 0 & 0 & -w^2 & \pi_\xi(u,v) \\ \pi_\xi(u,v) & -w & 0 & 0 \\ w^2 & u-\xi v & 0 & 0 \end{pmatrix} \hat{\otimes} Q_z \oplus \Big(v \leftrightarrow w \Big) \, . \tag{C.6}$$

The equivariance map ω can then be chosen to permute the two factorisations in the superposition. Before orbifolding we have four bosons and four fermions coming from each of the two self-spectra, and four bosons and four fermions from the relative overlap, giving in total 12 bosons and 12 fermions. Half of those are even under the $\mathbb{Z}_2$-action, so we find 6 bosons and 6 fermions in the orbifold. This agrees with the results of (4.7).

This begs the question of what the $P(j)$-branes of the $A_1 \times A_4$ model correspond to in the $A_1 \times A_1 \times A_1'$ theory. We propose that they correspond to the factorisation

$$Q_P = \begin{pmatrix} 0 & E_P \\ J_P & 0 \end{pmatrix} \hat{\otimes} Q_z \tag{C.7}$$

with

$$E_P = \begin{pmatrix} \pi_\xi(u',v) & u'-\xi w \\ \pi_\xi(u',w) & -(u'-\xi v) \end{pmatrix} \qquad J_P = \begin{pmatrix} u'-\xi v & u'-\xi w \\ \pi_\xi(u',w) & -\pi_\xi(u',v) \end{pmatrix} \, . \tag{C.8}$$

Here, we have rescaled u to $u' = u/\sqrt[3]{2}$, and the three different values of j correspond to the three different values of ξ. The representation ω of the orbifold group is

$$\omega = \begin{pmatrix} \omega_0 & 0 \\ 0 & \omega_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, , \qquad \text{with} \qquad \omega_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, , \qquad \omega_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, . \tag{C.9}$$

In order to support this claim we shall now determine the self-spectrum of this factorisation.

C.1.1 The spectrum of the P-factorisation

First we note that Q_P has the structure of a tensor product of a permutation factorisation of the $(u'v)$- and one of the $(u'w)$-part. The spectrum, however, is not just the product of the two spectra, because the variables involved in the two factorisations are not independent. To begin with, let us ignore the z-part of Q_P. A closed boson

$$\phi = \begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{pmatrix} \tag{C.10}$$

satisfies $E_P\phi_1 = \phi_0 E_P$ and $J_P\phi_0 = \phi_1 J_P$. By adding exact morphisms, it can always be brought to the form

$$\phi_0 = \phi_1 = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \tag{C.11}$$

where s is a polynomial in u', v, w. The remaining freedom of adding exact morphisms means that s is defined up to elements in the ideal generated by $(u' - \xi v)$, $(u' - \xi w)$, $\pi_\xi(u', v)$ and $\pi_\xi(u', w)$. We can easily see that two bosons remain, both of which are even under the action of the orbifold group.
A closed fermion

$$\psi = \begin{pmatrix} 0 & \psi_1 \\ \psi_0 & 0 \end{pmatrix} \tag{C.12}$$

satisfies $E_P\psi_0 + \psi_1 J_P = 0$ and $J_P\psi_1 + \psi_0 E_P = 0$. By adding exact morphisms, it can always be brought to the form

$$\psi_0 = \begin{pmatrix} t & -t \\ t(u' + \xi v + \xi w) & t(u' + \xi v + \xi w) \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} -t(u' + \xi v + \xi w) & -t \\ t(u' + \xi v + \xi w) & -t \end{pmatrix}. \tag{C.13}$$

The remaining freedom of adding exact morphisms means that the polynomial t is defined up to elements in the ideal generated by $(u' - \xi v)$, $(u' - \xi w)$ and $(v^2 + vw + w^2)$. There are two fermions that remain which can be represented by $t = 1$ and $t = v + w$. Under the orbifold action, these two fermions transform as

$$\psi(u, v, w) \to \omega\psi(u, w, v)\omega^{-1} = -\psi(u, v, w)\,, \tag{C.14}$$

so they are odd under the $\mathbb{Z}_2$-action.

Tensoring the spectrum together with the even boson and the odd fermion coming from the z-part, we find four bosons and no fermions in the orbifolded theory. This then agrees with the open string spectrum on the generalised permutation factorisation of the $A_1 \times A_4$-model.

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